Proof of absence of local conserved quantity in some nonintegrable models

Naoto Shiraishi (Gakushuin university)

N. Shiraishi, Europhys. Lett. 128 17002 (2019)

Outline

- Background
- Proof (case of 3-support)
- Proof (general case)
- Extension

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- **Background**
- Proof (case of 3-support)
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Integrable systems and non-integrable systems

Integrable systems

- Many models are proven to be integrable.
- Various techniques are developed.
- A little artificial (e.g., no thermalization)

Non-integrable systems

- Almost all natural systems are considered to be non-integrable.
- **No concrete model is proven to be non-integrable!**

Local conserved quantity

We employ **absence of local conserved quantity** as a working definition of non-integrability.

Def：local conserved quantity (LCQ)

Conserved quantity given by sum of local quantities

We can show presence of LCQ in some systems.
Can we show **absence** of LCQ in a concrete system?

Main result

Theorem

S=1/2 XYZ chain with z magnetic field (with p.b.c.)

$$H = \sum_i J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z + h S_i^z$$

has no nontrivial LCQ if $J_x, J_y, J_z \neq 0$, $J_x \neq J_y$ and $h \neq 0$.

(N. Shiraishi, Europhys. Lett. 128 17002 (2019))

Other examples

- Heisenberg model with next-nearest interaction
- Heisenberg model with staggered magnetic field

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- **Proof (case of 3-support)**
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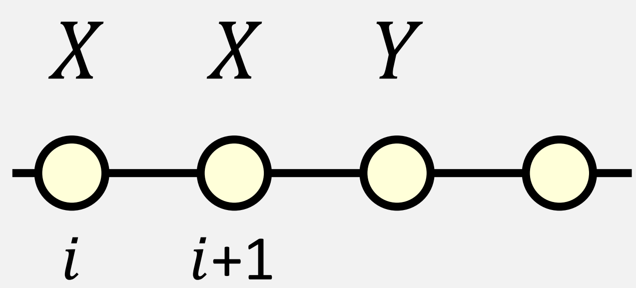
Symbols in this talk (1)

Q : candidate of conserved quantity (CQ)

Q is shift invariant because H is shift invariant

Ex) $$Q = \sum_i 2X_iX_{i+1}X_{i+2} - 3X_iX_{i+1}Y_{i+2} + \cdots$$

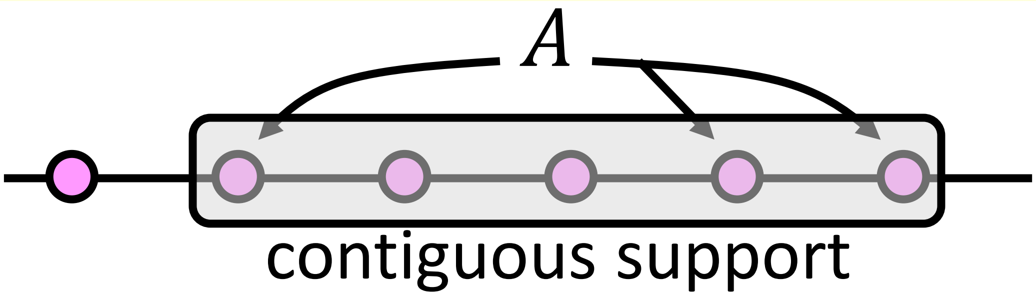
We denote $\sum_i$



by XXY

k-support conserved quantity

k-support operator: Shift sum of operators supported by k contiguous sites.



k-support CQ: Conserved quantity which consists of at most k-support operators.

Approach for proof：Proving absence of k-support CQ from small k.

What we should do (for $k = 3$)

Candidate of 3-support CQ

$$Q = \sum_i q_{XXX}XXX + q_{XXY}XXY + \cdots + q_{ZIZ}ZIZ$$
$$\underbrace{+q_{XX}XX + q_{XY}XY + \cdots + q_Y Y + q_Z Z}_{}$$

at most 64 terms!

Theorem (for $k = 3$)

If $[\boldsymbol{Q}, \boldsymbol{H}] = \boldsymbol{0}$, then $\boldsymbol{q_{ABC}} = \boldsymbol{0}$ for any ABC.

Symbols in this talk (2)

$$-i[X_iY_{i+1}Z_{i+2}, X_{i+2}X_{i+3}] = 2X_iY_{i+1}Y_{i+2}X_{i+3},$$

Denote it as follows (including multiplication of $-i$)

$$\begin{array}{ccccc} & X & Y & Z & \\ & & & X & X \\ \hline 2 & X & Y & Y & X \end{array}$$

Symbols in this talk (2)

$$-i[X_iY_{i+1}Z_{i+2}, X_{i+2}X_{i+3}] = 2X_iY_{i+1}Y_{i+2}X_{i+3},$$

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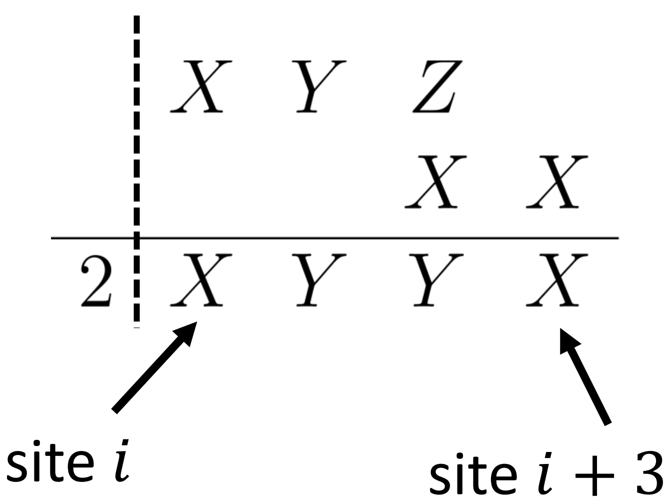
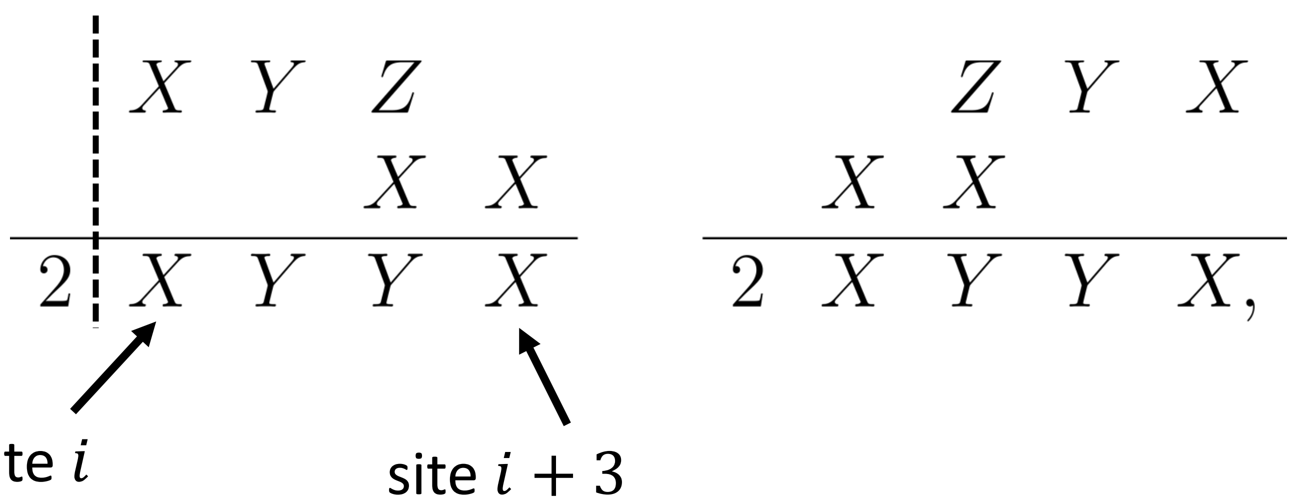


Symbols in this talk (2)

$$-i[X_i Y_{i+1} Z_{i+2}, X_{i+2} X_{i+3}] = 2 X_i Y_{i+1} Y_{i+2} X_{i+3},$$
$$-i[Z_{i+1} Y_{i+2} X_{i+3}, X_i X_{i+1}] = 2 X_i Y_{i+1} Y_{i+2} X_{i+3}.$$

Denote it as follows (including multiplication of $-i$)



Relation between coefficients in Q

$$\begin{array}{ccccc} & X & Y & Z & \\ & & & X & X \\ \hline 2 & X & Y & Y & X \end{array} \qquad \begin{array}{ccccc} & & Z & Y & X \\ & X & X & & \\ \hline 2 & X & Y & Y & X, \end{array}$$

Because coefficient of $XYYX$ in $[Q, H]$ is zero,

$$q_{XYZ} + q_{ZYX} = 0$$

Some coefficients are zero

This is the unique commutation generating $XXXY$

$$\begin{array}{ccccc} & X & X & Z & \\ & & & Y & Y \\ \hline -2 & X & X & X & Y \end{array}$$

($\because$ no operator satisfies the following relation)

$$\begin{array}{ccccc} & & ? & ? & ? \\ & X & X & & \\ \hline \pm 2 & X & X & X & Y \end{array}$$

Some coefficients are zero

This is the unique commutation generating $XXXY$

$$\begin{array}{ccccc} & X & X & Z & \\ & & & Y & Y \\ \hline -2 & X & X & X & Y \end{array}$$

Because the coefficient of $XXXY$ in $[Q, H]$ is zero,

$$q_{XXZ} = 0$$

Connecting coefficient to another coefficient known to be zero

Since coefficient of XXZ is zero, coefficient of operators "pairing with" XXZ is also zero.

$$
\begin{array}{ccccc}
 & Y & Z & Y & \\
 & & & Z & Z \\
\hline
2 & Y & Z & X & Z
\end{array}
\qquad
\begin{array}{ccccc}
 & & X & X & Z \\
 & Y & Y & & \\
\hline
2 & Y & Z & X & Z,
\end{array}
$$

$$\rightarrow \quad J_Z q_{YZY} + J_Y q_{XXZ} = 0$$

$$\rightarrow \quad J_Z q_{YZY} = -J_Y q_{XXZ} = 0$$

Consequence from consideration of 4-support operators

q_{ABC} is zero if

- two of A, B, C are the same, or
- $B = I$

Coefficients which might be nonzero are only

$$J_X q_{YXZ} = J_Y q_{ZYX} = J_Z q_{XZY}$$
$$= - J_X q_{ZXY} = -J_Y q_{XYZ} = -J_Z q_{YZX}.$$

It suffices to prove one of them is zero!

Analysis on 3-support operators

Consider YZY in $[Q, H]$.

$$
\begin{array}{cccc}
 & Y & Z & X \\
 & & & Z \\
\hline
-2 & Y & Z & Y
\end{array}
\quad
\begin{array}{cccc}
 & X & Z & Y \\
 & Z & & \\
\hline
-2 & Y & Z & Y
\end{array}
\quad
\begin{array}{cccc}
 & Y & X & \\
 & & Y & Y \\
\hline
2 & Y & Z & Y
\end{array}
\quad
\begin{array}{cccc}
 & & X & Y \\
 & Y & Y & \\
\hline
2 & Y & Z & Y,
\end{array}
$$

(single Z comes from z magnetic field in Hamiltonian)

Usually, 4 types of commutators "generate" a single 3-support operator

$$\rightarrow \quad h(q_{YZX} + q_{XZY}) - J_Y(q_{YX} + q_{XY}) = 0$$

Operators generated by only 3 types of commutators

Some 3-support operators are generated by 3 types of commutators in $[Q, H]$.

$$\begin{array}{cccc} & X & Y & Z \\ & Z & & \\ \hline -2 & Y & Y & Z \end{array} \qquad \begin{array}{cccc} & Y & X & Z \\ & & Z & \\ \hline -2 & Y & Y & Z \end{array} \qquad \begin{array}{cccc} & Y & X & \\ & & Z & Z \\ \hline -2 & Y & Y & Z \end{array}$$

∵ No 2-support operator in Q satisfies

$$\begin{array}{cccc} & & ? & ? \\ & Y & Y & \\ \hline -2 & Y & Y & Z \end{array}$$

Operators generated by only 3 types of commutators

Some 3-support operators are generated by 3 types of commutators in $[Q, H]$.

$$
\begin{array}{cccc} & X & Y & Z \\ & Z & & \\ \hline -2 & Y & Y & Z \end{array}
\qquad
\begin{array}{cccc} & Y & X & Z \\ & & Z & \\ \hline -2 & Y & Y & Z \end{array}
\qquad
\begin{array}{cccc} & Y & X & \\ & & Z & Z \\ \hline -2 & Y & Y & Z \end{array}
$$

$$h(q_{XYZ} + q_{YXZ}) + J_Z q_{YX} = 0$$

$$
\begin{array}{cccc} & X & Y & Z \\ & & Z & \\ \hline 2 & X & X & Z \end{array}
\qquad
\begin{array}{cccc} & Y & X & Z \\ & Z & & \\ \hline 2 & X & X & Z \end{array}
\qquad
\begin{array}{cccc} & X & Y & \\ & & Z & Z \\ \hline 2 & X & X & Z, \end{array}
$$

$$h(q_{XYZ} + q_{YXZ}) + J_Z q_{XY} = 0$$

Relations between coefficients

The obtained three equalities

$$h(q_{XYZ} + q_{YXZ}) + J_Z q_{XY} = 0$$
$$h(q_{YZX} + q_{XZY}) - J_Y(q_{YX} + q_{XY}) = 0$$
$$h(q_{XYZ} + q_{YXZ}) + J_Z q_{YX} = 0$$

By erasing q_{XY}, q_{YX}, and using

$$J_X q_{YXZ} = J_Y q_{ZYX} = J_Z q_{XZY}$$
$$= - J_X q_{ZXY} = -J_Y q_{XYZ} = -J_Z q_{YZX},$$

Result (for $k = 3$)

$$3h\left(1 - \frac{J_Y}{J_X}\right) q_{XYZ} = 0$$

Unless $h = 0$ (XYZ model) or $J_X = J_Y$ (XXZ model with a z-magnetic field), we have $\boldsymbol{q_{XYZ} = 0}$

→**Absence of 3-support conserved quantity!**

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- **Proof (general case)**
- Extension

logic flow for $k = 3$

<u>Analysis on 4-support operators in $[Q, H]$</u>

- Restrict a candidate of 3-support CQ in a specific form
- Derive linear relation between coefficients

<u>Analysis on 3-support operators in $[Q, H]$</u>

- Demonstrating coefficient of one of remaining 3-support operators equal to zero.

Logic flow for general k is almost the same!

logic flow for general k

<u>Analysis on k+1-support operators in $[Q, H]$</u>

- Restrict a candidate of k-support CQ in a specific form
- Derive linear relation between coefficients

<u>Analysis on k-support operators in $[Q, H]$</u>

- Demonstrating coefficient of one of remaining k-support operators equal to zero.

logic flow for general k

Analysis on k+1-support operators in $[\boldsymbol{Q}, \boldsymbol{H}]$

- **Restrict a candidate of k-support CQ in a specific form**
- **Derive linear relation between coefficients**

Analysis on k-support operators in $[Q, H]$

- Demonstrating coefficient of one of remaining k-support operators equal to zero.

Which coefficients are connected in analysis on k+1-support operators

$$\begin{array}{rcccccc} & Y & Z & Z & Z & X & \\ & & & & & Z & Z \\ \hline -2 & Y & Z & Z & Z & Y & Z \end{array} \qquad \begin{array}{rcccccc} & & X & Z & Z & Y & Z \\ & Y & Y & & & & \\ \hline 2 & Y & Z & Z & Z & Y & Z, \end{array}$$

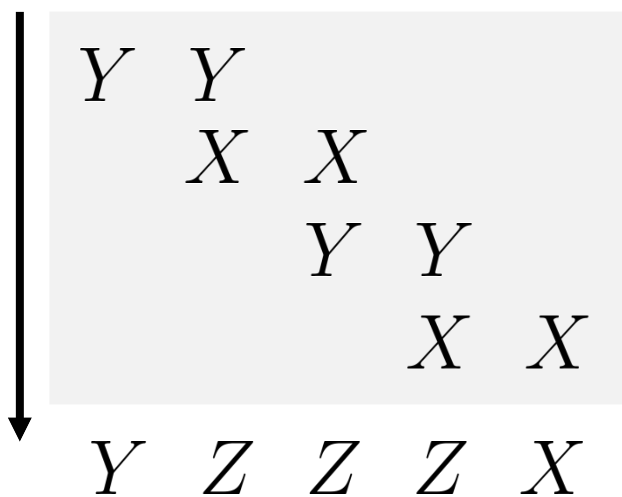
Coefficient of $YZZZX$ and that of $XZZYZ$ are in linear relation.

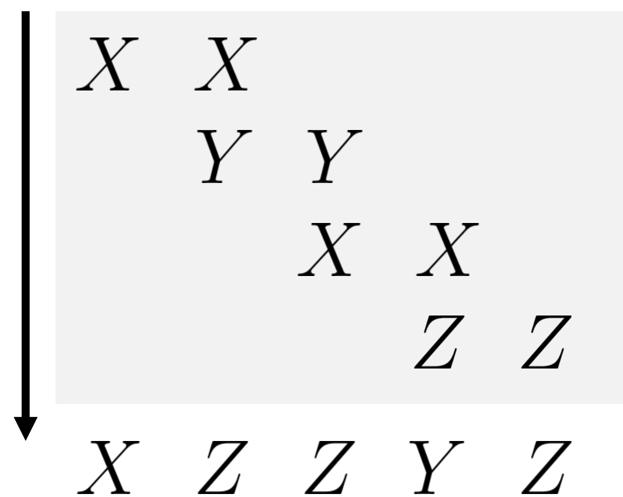
Why $YZZZX$ and $XZZYZ$ are connected?
What property lies in these two operators?

Signless product

<u>Signless product of Pauli matrices</u>

$$XY = YX = Z,\ XZ = ZX = Y,\ YZ = ZY = X$$

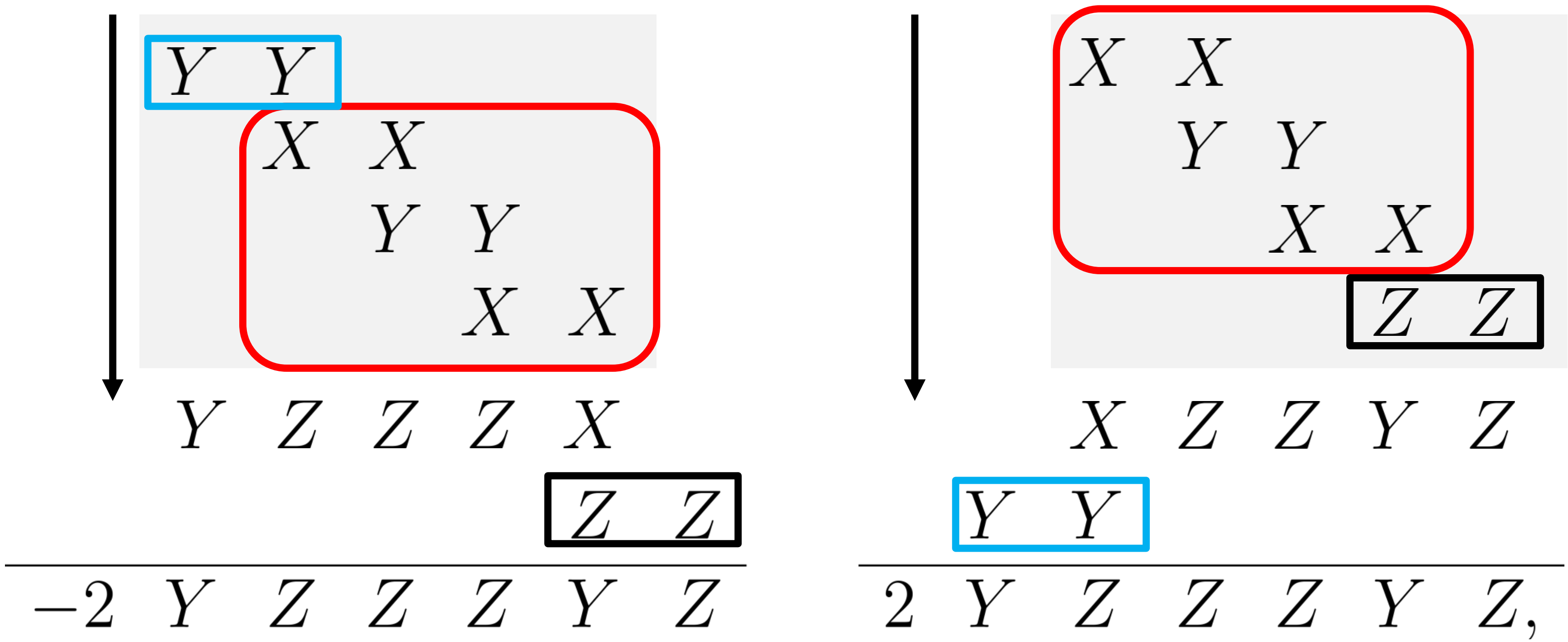




Doubling operator: XX, YY, ZZ

Doubling-product: operators expressed as above.

Why these two are connected?



By removing YY from left, and adding ZZ to right,
$YZZZX$ becomes $XZZYZ$.

Connected operators through commutation relations

Connection between operators:
→Adding/removing doubling operators (XX, YY, ZZ) at the left/right end

We then find…

Doubling-product : maybe nonzero coefficient
Not doubling-product : zero coefficient!

Case of 3-support operators (revisit)

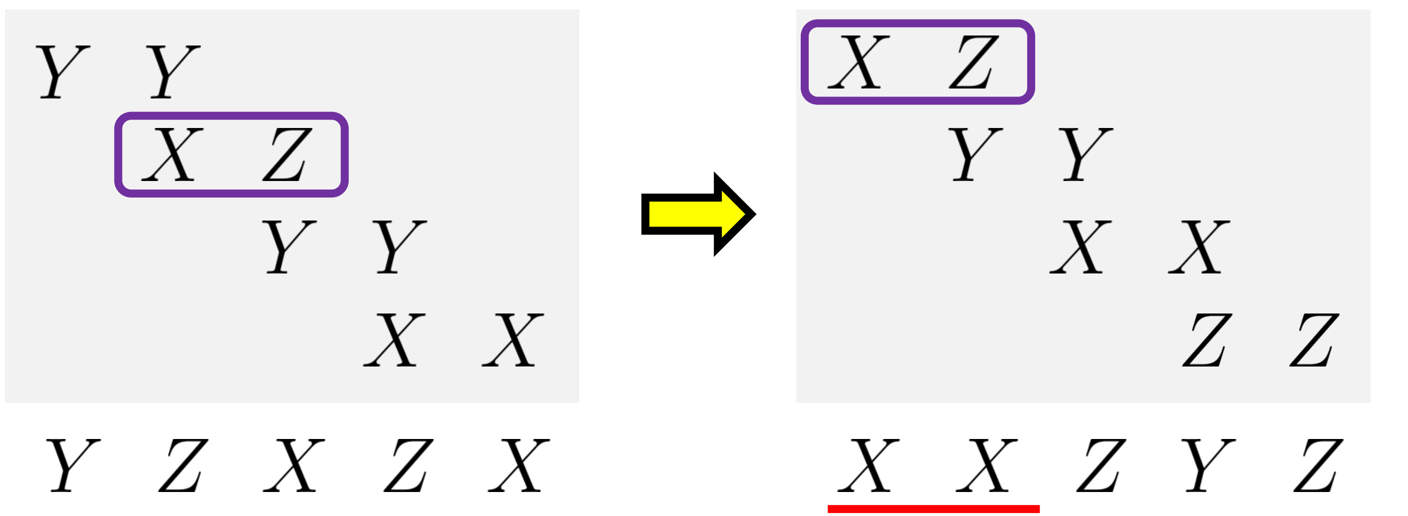
Coefficients which might be nonzero are

$$J_X q_{YXZ} = J_Y q_{ZYX} = J_Z q_{XZY}$$
$$= - J_X q_{ZXY} = -J_Y q_{XYZ} = -J_Z q_{YZX},$$

Ex) YZX can be expressed as

$$\downarrow \begin{array}{ccc} Y & Y & \\ & X & X \\ \hline Y & Z & X \end{array}$$

How coefficient of non-doubling-product operator vanishes?



Two leftmost operators are the same!

→No operator make pair with $XXZYZ$

→Coefficient should be zero!

Consequence of analysis on k+1-support operators

Analysis on k+1-support operators in $[Q, H]$

- Restrict a candidate of k+1-support CQ in a specific form

→**Non-doubling-product operator has zero coefficient.**

- Derive linear relation between coefficients

→**Coefficients of doubling-product operators are obviously in linear relation.**

logic flow for general k

Analysis on k+1-support operators in $[Q, H]$

- Restrict a candidate of k+1-support CQ in a specific form
- Derive linear relation between coefficients

Analysis on k-support operators in $[\boldsymbol{Q}, \boldsymbol{H}]$

- **Demonstrating coefficient of one of remaining k-support operators equal to zero.**

Analysis on 3-support operators (revisit)

Consider YZY in $[Q,H]$.

$$\begin{array}{cccc} & Y & Z & X \\ & & & Z \\ \hline -2 & Y & Z & Y \end{array} \qquad \begin{array}{cccc} & X & Z & Y \\ & Z & & \\ \hline -2 & Y & Z & Y \end{array}$$

$$\begin{array}{cccc} & Y & X & \\ & & Y & Y \\ \hline 2 & Y & Z & Y \end{array} \qquad \begin{array}{cccc} & & X & Y \\ & Y & Y & \\ \hline 2 & Y & Z & Y, \end{array}$$

3-support in Q
1-support in H

↓

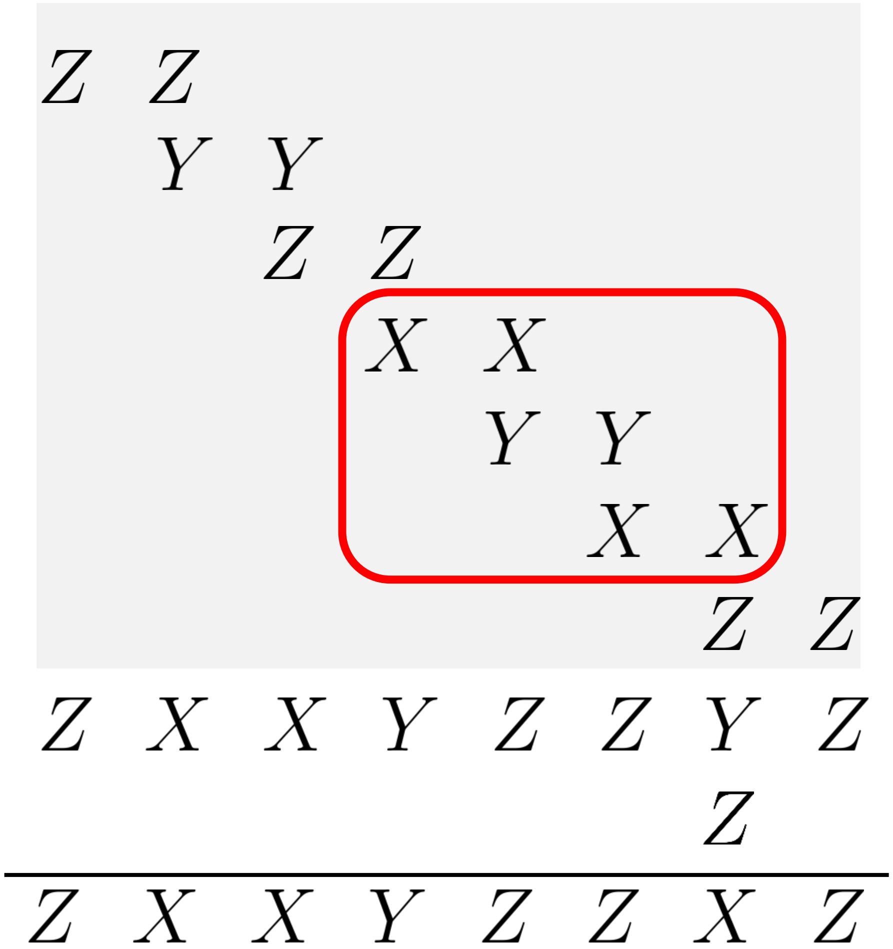
k-support in Q
1-support in H

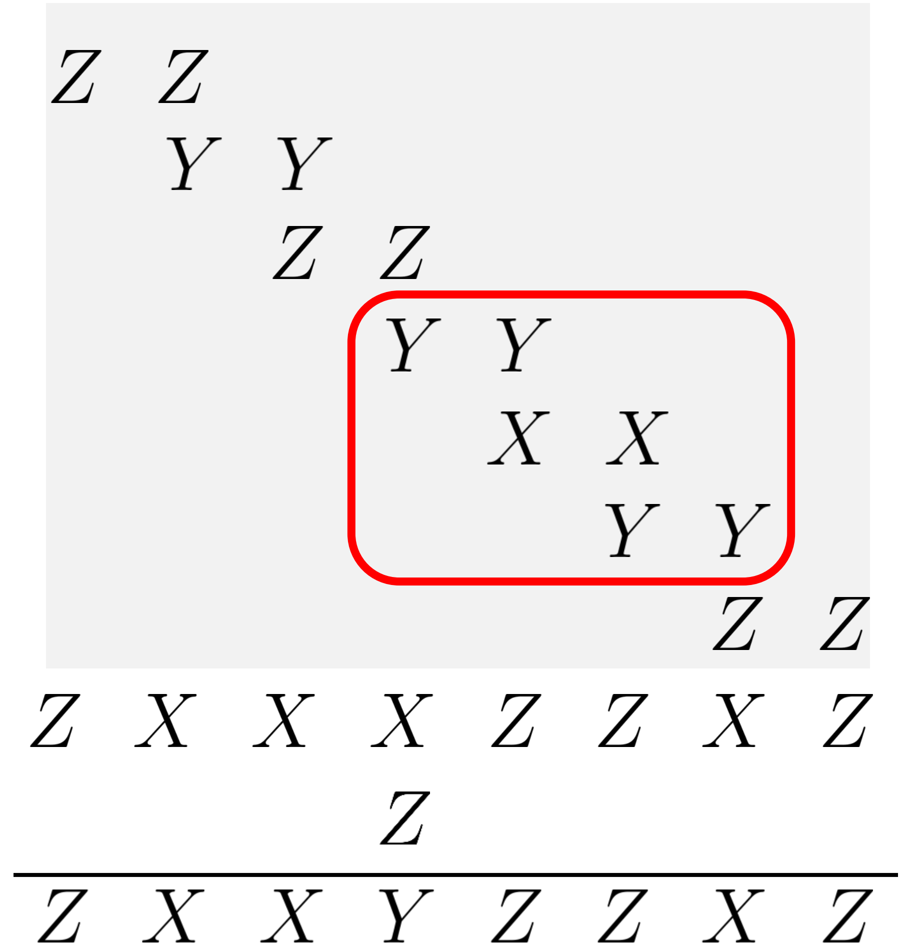
2-support in Q
2-support in H

↓

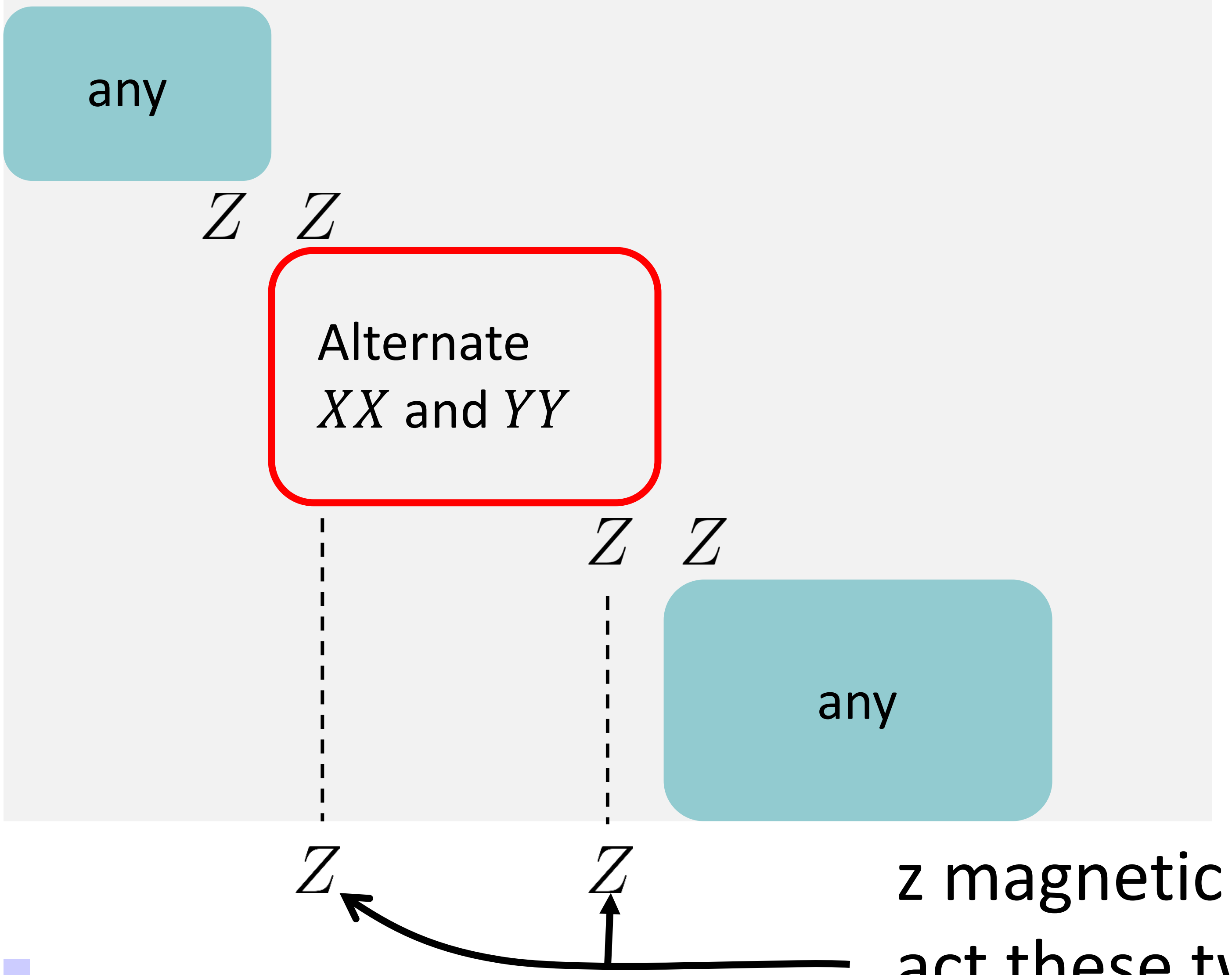
k-1-support in Q
2-support in H

Where can z magnetic field act?



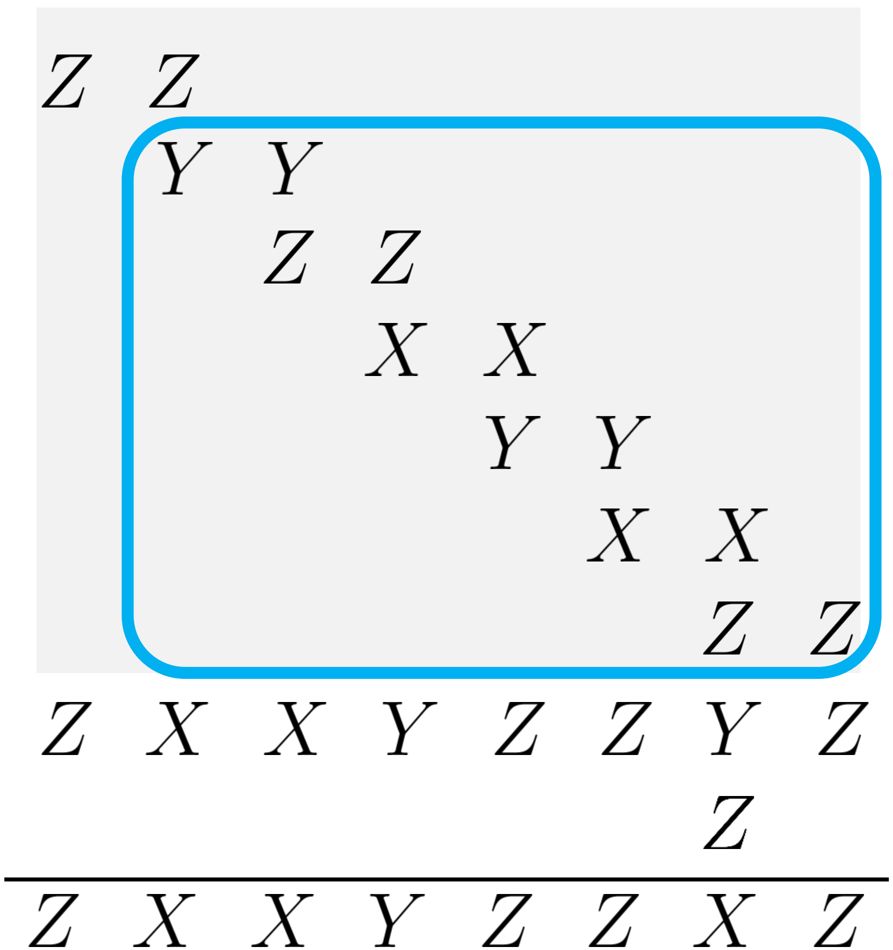


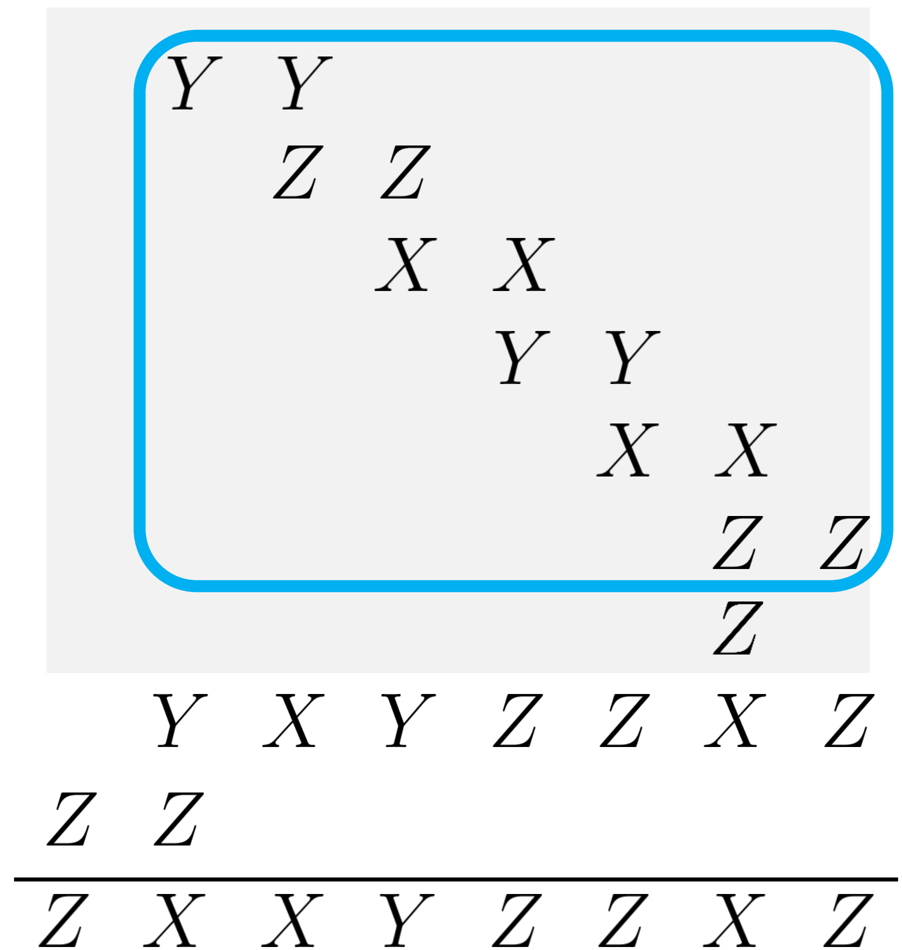
Where can z magnetic field act?



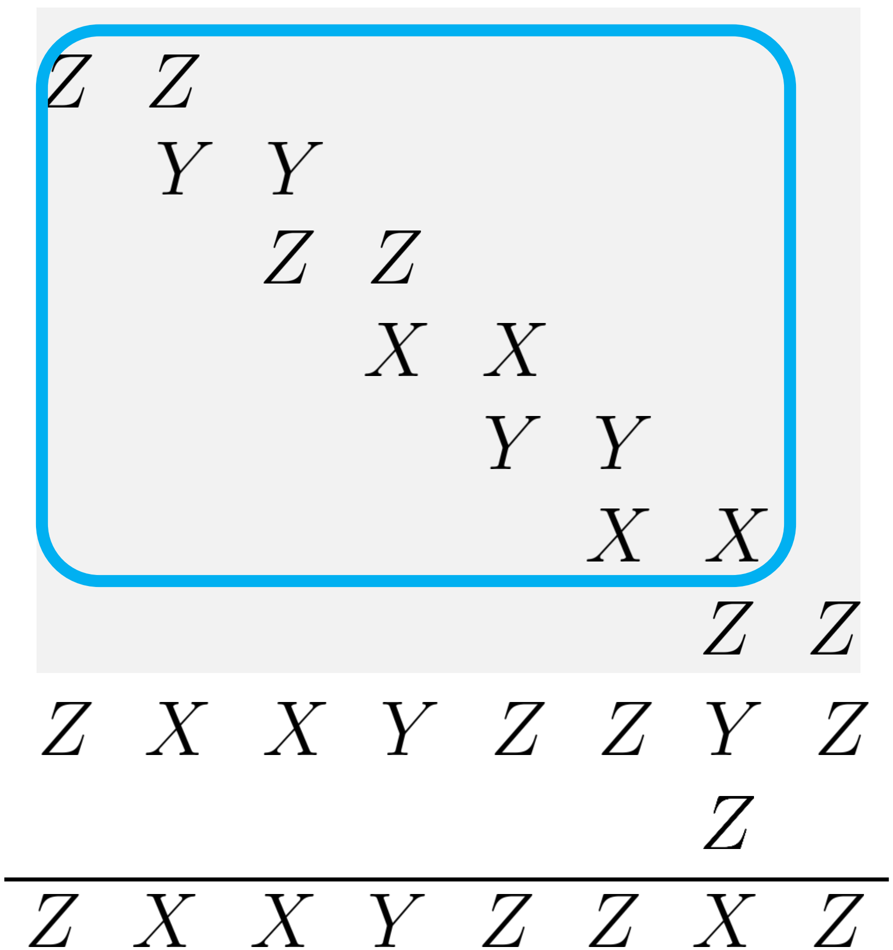
z magnetic field can act these two sites.

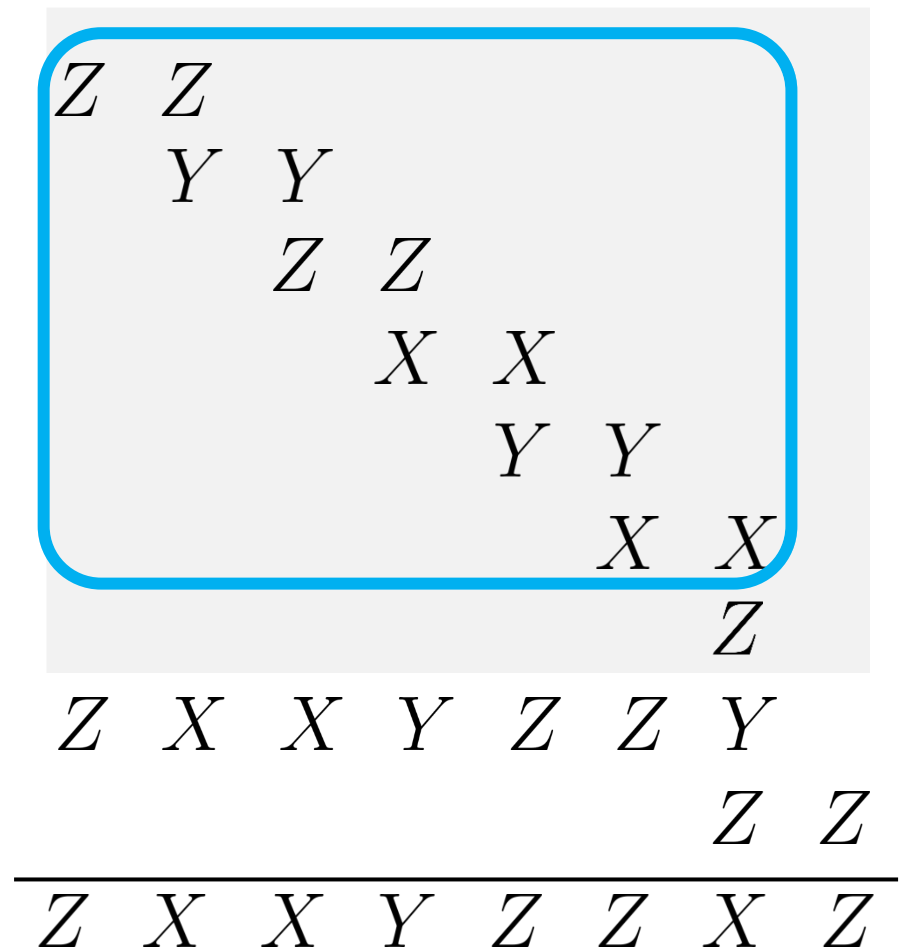
Corresponding k-1-support + 2-support





Corresponding k-1-support + 2-support





Analysis on 3-support operators (revisit)

Some 3-support operators are generated by 3 types of commutators in $[Q, H]$.

$$\begin{array}{cccc} & X & Y & Z \\ & Z & & \\ \hline -2 & Y & Y & Z \end{array} \qquad \begin{array}{cccc} & Y & X & Z \\ & & Z & \\ \hline -2 & Y & Y & Z \end{array} \qquad \begin{array}{cccc} & Y & X & \\ & & Z & Z \\ \hline -2 & Y & Y & Z \end{array}$$

3-support in Q
1-support in H

↓

k-support in Q
1-support in H

2-support in Q
2-support in H

↓

k-1-support in Q
2-support in H

k-support operator generated by one type of k-1-support + 2-support

If left/right end of two operators are the same, 2-support operator in H cannot act (ex: $k = 6$).

$$
\begin{array}{cccccc}
Z & Z & & & & \\
 & X & X & & & \\
 & & Y & Y & & \\
 & & & Z & Z & \\
 & & & & X & X \\
Z & Y & Z & X & Y & X \\
 & & & & Z & \\
\hline
Z & Y & Z & X & \boxed{X} & \boxed{X}
\end{array}
\qquad
\begin{array}{cccccc}
Z & Z & & & & \\
 & X & X & & & \\
 & & Y & Y & & \\
 & & & Z & Z & \\
 & & & & Y & Y \\
Z & Y & Z & X & X & Y \\
 & & & & & Z \\
\hline
Z & Y & Z & X & \boxed{X} & \boxed{X}
\end{array}
$$

k-support operator generated by one type of k-1-support + 2-support

If left/right end of two operators are the same, 2-support operator in H cannot act (ex: $k = 6$).

$$
\begin{array}{cccccc}
Z & Z & & & & \\
 & X & X & & & \\
 & & Y & Y & & \\
 & & & Z & Z & \\
 & & & & X & X \\
Z & Y & Z & X & Y & X \\
 & & & & Z & \\
\hline
Z & Y & Z & X & \boxed{X} & \boxed{X}
\end{array}
\qquad
\begin{array}{cccccc}
? & ? & ? & ? & ? & \\
 & & & & X & X \\
\hline
Z & Y & Z & X & \boxed{X} & \boxed{X}
\end{array}
$$

k-support operator generated by one type of k-1-support + 2-support

If left/right end of two operators are the same, 2-support operator in H cannot act (ex: $k=6$).

Z	Z				
	X	X			
		Y	Y		
			Z	Z	
				X	X
Z	Y	Z	X	Y	X
				Z	
Z	Y	Z	X	X	X

Only single commutation generates this operator if

- Ordering XX (or YY), ZZ from left/right end
- z act at left/rightmost or next to left/rightmost.

Case of 3-support operators (revisit)

$$
\begin{aligned}
h(q_{XYZ} + q_{YXZ}) + \qquad\qquad\qquad\qquad J_Z q_{XY} &= 0 \\
h(q_{YZX} + q_{XZY}) - J_Y(q_{YX} + q_{XY}) &= 0 \\
h(q_{XYZ} + q_{YXZ}) + J_Z q_{YX} \qquad\qquad\qquad &= 0
\end{aligned}
$$

Case of 3-support operators (revisit)

k-support | k-1-support

$$h(q_{XYZ} + q_{YXZ}) + J_Z q_{XY} = 0$$

$$h(q_{YZX} + q_{XZY}) - J_Y (q_{YX} + q_{XY}) = 0$$

$$h(q_{XYZ} + q_{YXZ}) + J_Z q_{YX} = 0$$

Case of 3-support operators (revisit)

k-support | k-1-support

$$h(q_{XYZ} + q_{YXZ}) + J_Z q_{XY} = 0$$
$$h(q_{YZX} + q_{XZY}) - J_Y(q_{YX} + q_{XY}) = 0$$
$$h(q_{XYZ} + q_{YXZ}) + J_Z q_{YX} = 0$$

Case of 3-support operators (revisit)

k-support | k-1-support

$$h(q_{XYZ}+q_{YXZ})+J_Z q_{XY}=0$$

$$h(q_{YZX}+q_{XZY})-J_Y(q_{YX}+q_{XY})=0$$

$$h(q_{XYZ}+q_{YXZ})+J_Z q_{YX}=0$$

Case of 3-support operators (revisit)

k-support   k-1-support

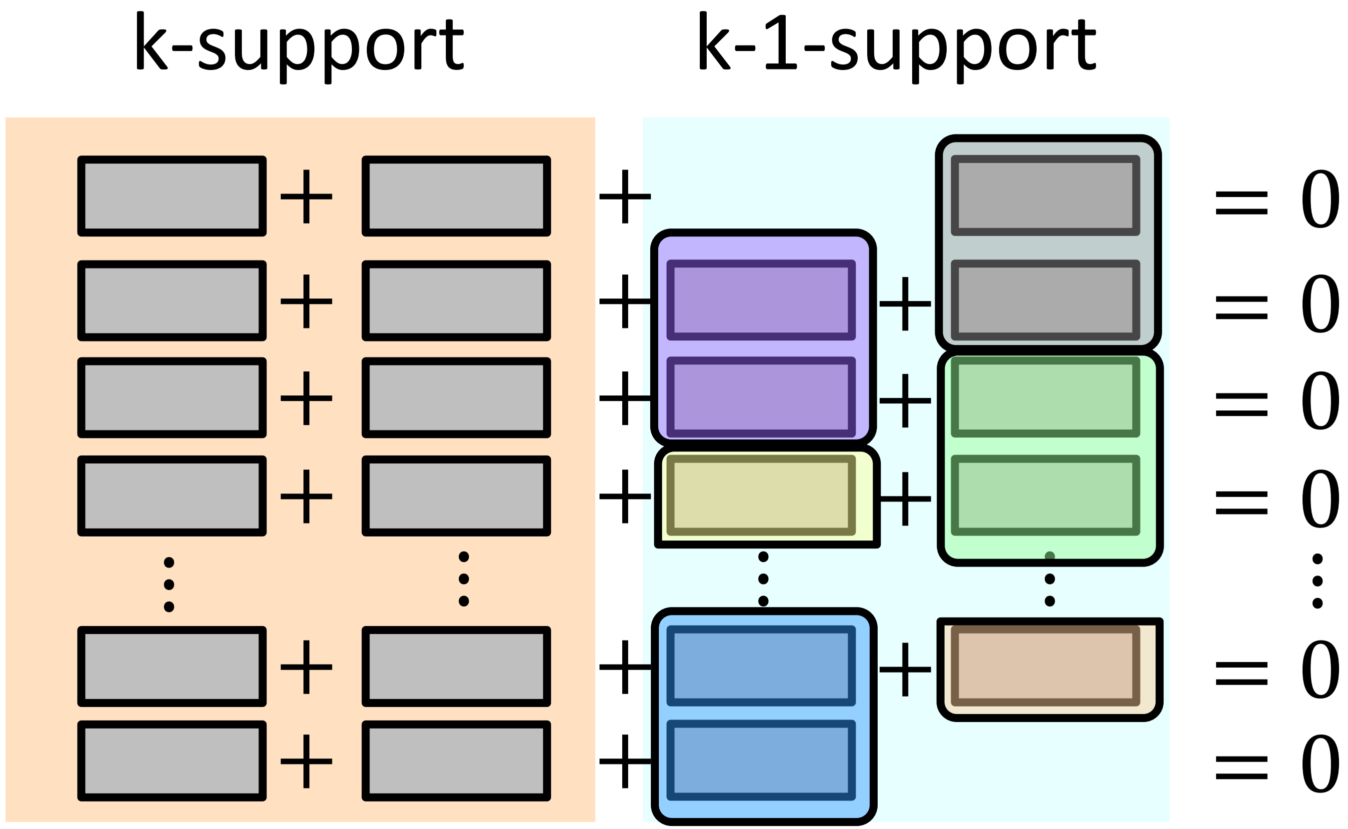
$$h(q_{XYZ} + q_{YXZ}) + J_Z q_{XY} = 0$$

$$h(q_{YZX} + q_{XZY}) - J_Y (q_{YX} + q_{XY}) = 0$$

$$h(q_{XYZ} + q_{YXZ}) + J_Z q_{YX} = 0$$

Same terms are canceled!

What we seek for case of k-support



Same terms are canceled!

The sequence

$$\underset{Z}{\uparrow}\overline{YZXY}\cdots\overline{YXZXY}$$
$$\underset{Z}{\uparrow}\overline{XYZXY}\cdots\overline{YXZX}$$
$$\underset{Z}{\uparrow}\overline{YXYZXY}\cdots\overline{YXZ}$$
$$\overline{Z}\underset{Z}{\uparrow}\overline{YZYZXY}\cdots\overline{YX}$$
$$\vdots$$
$$\vdots$$
$$\overline{XY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{YXYZ}$$
$$\overline{ZXY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{YXY}$$
$$\overline{YZXY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{YX}$$
$$\overline{XYZXY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{Y}$$

$$\overline{X}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YXZXY}$$
$$\overline{YX}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YXZX}$$
$$\overline{XYX}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YXZ}$$
$$\overline{ZXYX}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YX}$$
$$\vdots$$
$$\vdots$$
$$\overline{XY}\cdots\overline{YXZXYX}\underset{Z}{\uparrow}\overline{Z}$$
$$\overline{ZXY}\cdots\overline{YXZXYX}\underset{Z}{\uparrow}$$
$$\overline{YZXY}\cdots\overline{YXZXY}\underset{Z}{\uparrow}$$
$$\overline{XYZXY}\cdots\overline{YXZX}\underset{Z}{\uparrow}$$

$$\overset{Z}{|}\overline{YZXY}\cdots\overline{YXZX}\overset{\leftarrow}{+}\overline{Y}$$
$$\overset{Z}{|}\overline{XYZXY}\cdots\overline{YXZ}\overset{\leftarrow}{+}\overline{X}$$
$$\overset{Z}{|}\overline{YXYZXY}\cdots\overline{YX}\overset{\leftarrow}{+}\overline{Z}$$
$$\overline{Z}\overset{Z}{|}\overline{YZYZXY}\cdots\overline{Y}\overset{\leftarrow}{+}\overline{X}$$
$$\vdots$$
$$\vdots$$
$$\overline{XY}\cdots\overline{YXZ}\overset{Z}{|}\overline{YXY}\overset{\leftarrow}{+}\overline{Z}$$
$$\overline{ZXY}\cdots\overline{YXZ}\overset{Z}{|}\overline{YX}\overset{\leftarrow}{+}\overline{Y}$$
$$\overline{YZXY}\cdots\overline{YXZ}\overset{Z}{|}\overline{Y}\overset{\leftarrow}{+}\overline{X}$$

$$\overline{Y}\overset{\rightarrow}{+}\overline{X}\overset{Z}{|}\overline{ZXY}\cdots\overline{YXZX}$$
$$\overline{X}\overset{\rightarrow}{+}\overline{YX}\overset{Z}{|}\overline{ZXY}\cdots\overline{YXZ}$$
$$\overline{Z}\overset{\rightarrow}{+}\overline{XYX}\overset{Z}{|}\overline{ZXY}\cdots\overline{YX}$$
$$\vdots$$
$$\vdots$$
$$\overline{X}\overset{\rightarrow}{+}\overline{Y}\cdots\overline{YXZXYX}\overset{Z}{|}\overline{Z}$$
$$\overline{Z}\overset{\rightarrow}{+}\overline{XY}\cdots\overline{YXZXYX}\overset{Z}{|}$$
$$\overline{Y}\overset{\rightarrow}{+}\overline{ZXY}\cdots\overline{YXZXY}\overset{Z}{|}$$
$$\overline{X}\overset{\rightarrow}{+}\overline{YZXY}\cdots\overline{YXZX}\overset{Z}{|}$$

New symbols

$\overline{X}$: Doubling operator XX (similar to YY, ZZ)
If ordered, we take one-site shift.

$\underset{Z}{\uparrow}$: Commutation with k-support operator.
Commutation with z magnetic field

$\overset{Z}{|}$: Using when construct k-1-support operator
Multiply Z at this position

$\overset{\rightarrow}{+}$: Commutation relation at this edge.

Examples

$$
\begin{array}{cccccc}
Z & Z & & & & \\
 & X & X & & & \\
 & & Y & Y & & \\
 & & & Z & Z & \\
 & & & & X & X \\
\hline
Z & Y & Z & X & Y & X
\end{array}
$$

$$\overline{ZXYZX}$$

Examples

$$
\begin{array}{cccccc}
Z & Z & & & & \\
 & X & X & & & \\
 & & Y & Y & & \\
 & & & Z & Z & \\
 & & & & X & X \\
Z & Y & Z & X & Y & X \\
 & & & & Z & \\
\hline
Z & Y & Z & X & X & X
\end{array}
$$

$$\overline{ZXYZ} \underset{Z}{\uparrow} \overline{X}$$

$$
\begin{array}{ccccc}
X & X & & & \\
 & Y & Y & & \\
 & & Z & Z & \\
 & & & X & X \\
 & & & Z & \\
X & Z & X & X & X
\end{array}
$$

$$\overline{XYZ} \overset{Z}{|} \overline{X}$$

Examples

$$
\begin{array}{cccccc}
Z & Z & & & & \\
 & X & X & & & \\
 & & Y & Y & & \\
 & & & Z & Z & \\
 & & & & X & X \\
Z & Y & Z & X & Y & X \\
 & & & & Z & \\
\hline
Z & Y & Z & X & X & X
\end{array}
$$

$$
\overline{ZXYZ} \underset{Z}{\uparrow} \overline{X}
$$

$$
\begin{array}{cccccc}
 & X & X & & & \\
 & & Y & Y & & \\
 & & & Z & Z & \\
 & & & & X & X \\
 & & & & Z & \\
 & X & Z & X & X & X \\
Z & Z & & & & \\
\hline
Z & Y & Z & X & X & X
\end{array}
$$

$$
\overline{Z} \overset{\rightarrow}{+} \overline{XYZ} \overset{Z}{|} \overline{X}
$$

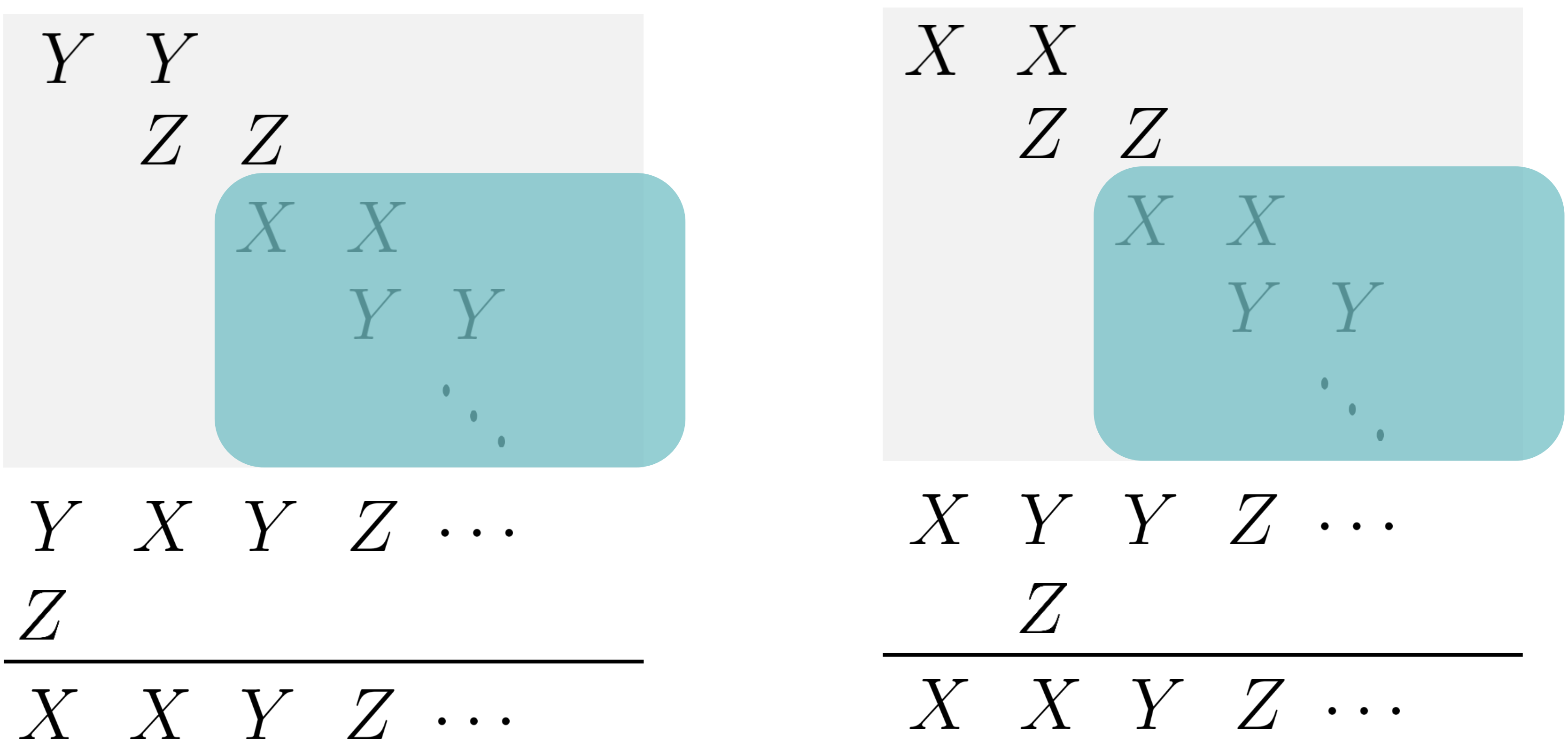
Structure

$$\begin{array}{llllll} Y & Y & & & & \\ & Z & Z & & & \\ & & X & X & & \\ & & & Y & Y & \\ & & & & \ddots & \end{array}$$

$$\frac{Y \quad X \quad Y \quad Z \quad \cdots \quad Z}{X \quad X \quad Y \quad Z \quad \cdots}$$

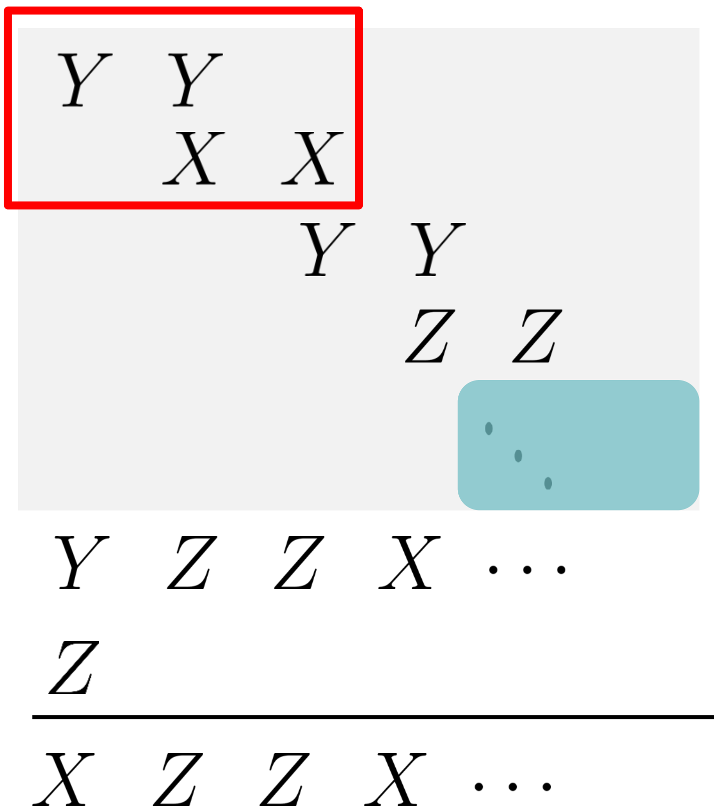
Details are not important

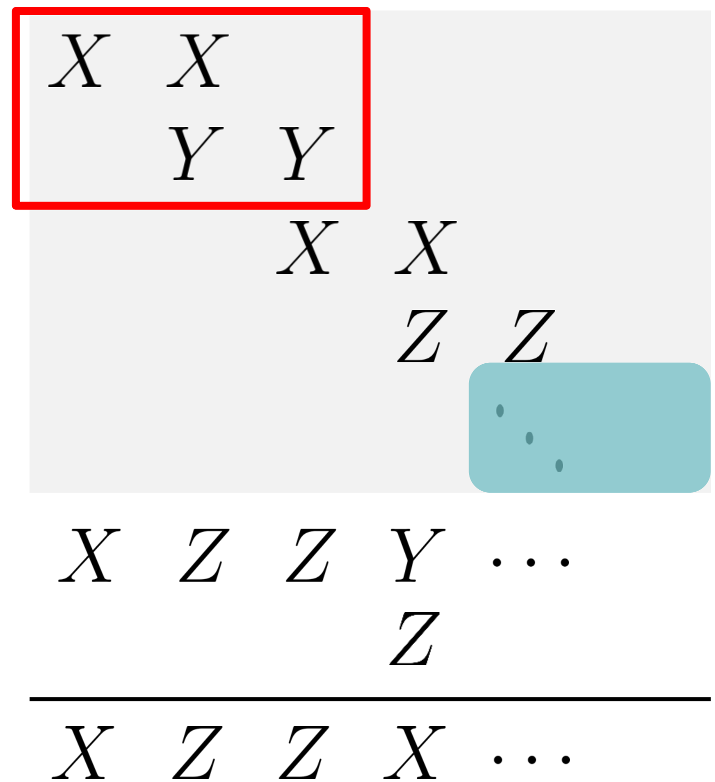
Structure

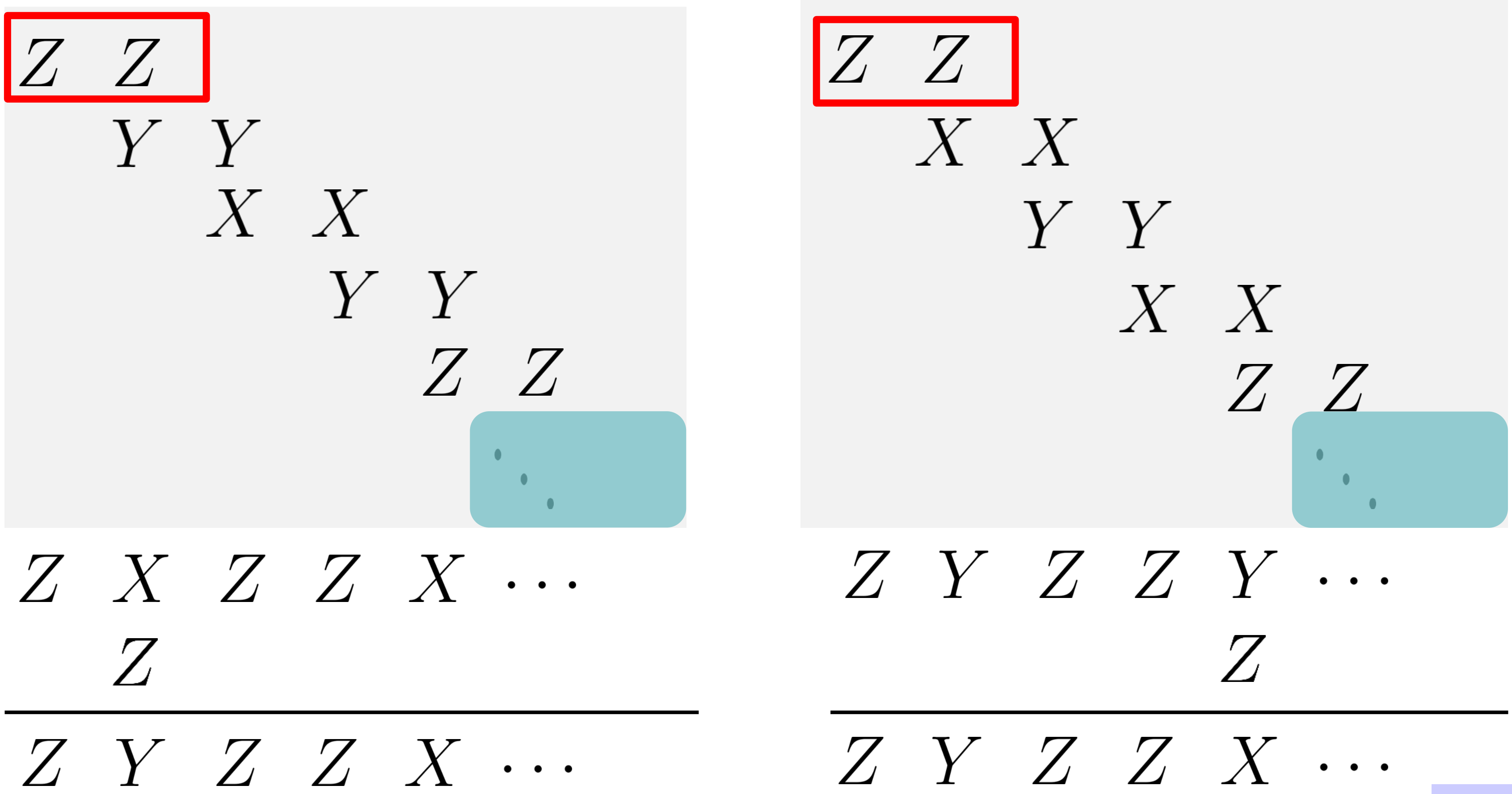


Since k-1-support operators are determined automatically, we hereafter omit them.

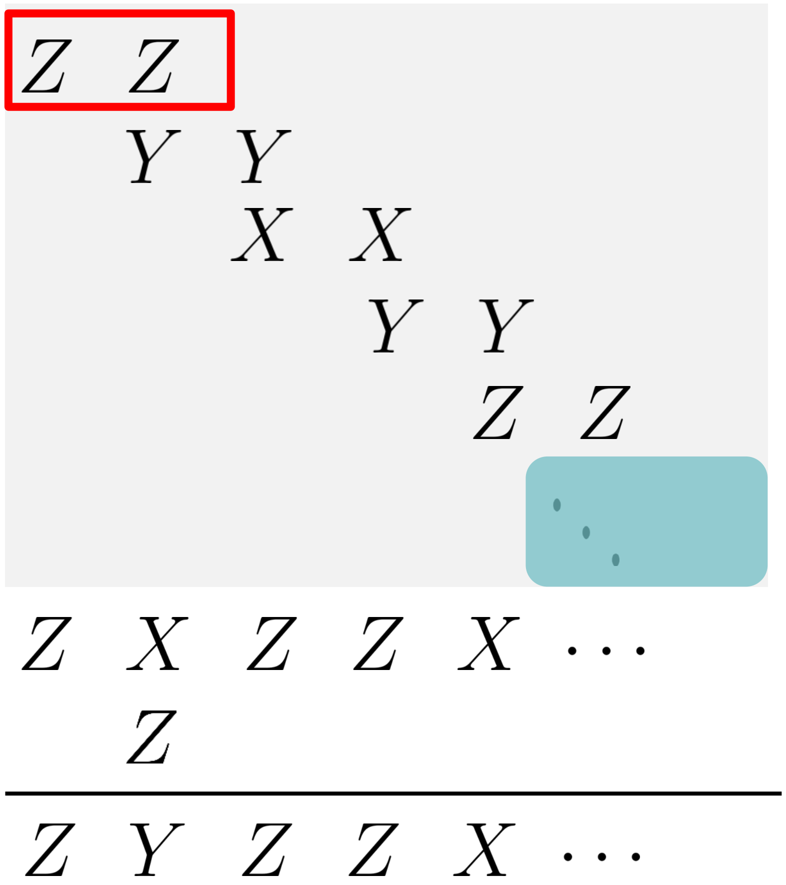
$\underset{Z}{\uparrow}\overline{YZXY}\cdots\overline{YXZXY}$	$\overline{X}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YXZXY}$	$\overset{Z}{\vert}\overline{YZXY}\cdots\overline{YXZX}\overset{\leftarrow}{+}\overline{Y}$	
$\underset{Z}{\uparrow}\overline{XYZXY}\cdots\overline{YXZX}$	$\overline{YX}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YXZX}$	$\overset{Z}{\vert}\overline{XYZXY}\cdots\overline{YXZ}\overset{\leftarrow}{+}\overline{X}$	$\overline{Y}\overset{\rightarrow}{+}\overline{X}\overset{Z}{\vert}\overline{ZXY}\cdots\overline{YXZX}$
$\underset{Z}{\uparrow}\overline{YXYZXY}\cdots\overline{YXZ}$	$\overline{XYX}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YXZ}$	$\overset{Z}{\vert}\overline{YXYZXY}\cdots\overline{YX}\overset{\leftarrow}{+}\overline{Z}$	$\overline{X}\overset{\rightarrow}{+}\overline{YX}\overset{Z}{\vert}\overline{ZXY}\cdots\overline{YXZ}$
$\overline{Z}\underset{Z}{\uparrow}\overline{YZYZXY}\cdots\overline{YX}$	$\overline{ZXYX}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YX}$	$\overline{Z}\overset{Z}{\vert}\overline{YZYZXY}\cdots\overline{Y}\overset{\leftarrow}{+}\overline{X}$	$\overline{Z}\overset{\rightarrow}{+}\overline{XYX}\overset{Z}{\vert}\overline{ZXY}\cdots\overline{YX}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\overline{XY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{YXYZ}$	$\overline{XY}\cdots\overline{YXZXYX}\underset{Z}{\uparrow}\overline{Z}$	$\overline{XY}\cdots\overline{YXZ}\overset{Z}{\vert}\overline{YXY}\overset{\leftarrow}{+}\overline{Z}$	$\overline{X}\overset{\rightarrow}{+}\overline{Y}\cdots\overline{YXZXYX}\overset{Z}{\vert}\overline{Z}$
$\overline{ZXY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{YXY}$	$\overline{ZXY}\cdots\overline{YXZXYX}\underset{Z}{\uparrow}$	$\overline{ZXY}\cdots\overline{YXZ}\overset{Z}{\vert}\overline{YX}\overset{\leftarrow}{+}\overline{Y}$	$\overline{Z}\overset{\rightarrow}{+}\overline{XY}\cdots\overline{YXZXYX}\overset{Z}{\vert}$
$\overline{YZXY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{YX}$	$\overline{YZXY}\cdots\overline{YXZXY}\underset{Z}{\uparrow}$	$\overline{YZXY}\cdots\overline{YXZ}\overset{Z}{\vert}\overline{Y}\overset{\leftarrow}{+}\overline{X}$	$\overline{Y}\overset{\rightarrow}{+}\overline{ZXY}\cdots\overline{YXZXY}\overset{Z}{\vert}$
$\overline{XYZXY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{Y}$	$\overline{XYZXY}\cdots\overline{YXZX}\underset{Z}{\uparrow}$		$\overline{X}\overset{\rightarrow}{+}\overline{YZXY}\cdots\overline{YXZX}\overset{Z}{\vert}$


Y Y
X X
Y Y
Z Z
Y Z Z X ...
Z
X Z Z X ...


X X
Y Y
X X
Z Z
X Z Z Y ...
Z
X Z Z X ...



Z Z
Y Y
X X
Y Y
Z Z
Z X Z Z X ...
Z
Z Y Z Z X ...
Z Z
X X
Y Y
X X
Z Z
Z Y Z Z Y ...
Z
Z Y Z Z X ...



- Inserting alternate XX, YY does not convey the leftmost Z.
- Inserting ZZ triggers to move Z to right.

$\underset{Z}{\uparrow}\overline{YZXY}\cdots\overline{YXZXY}$	$\overline{X}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YXZXY}$	$\overset{Z}{\vert}\overline{YZXY}\cdots\overline{YXZX}\overset{\leftarrow}{+}\overline{Y}$	
$\underset{Z}{\uparrow}\overline{XYZXY}\cdots\overline{YXZX}$	$\overline{YX}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YXZX}$	$\overset{Z}{\vert}\overline{XYZXY}\cdots\overline{YXZ}\overset{\leftarrow}{+}\overline{X}$	$\overline{Y}\overset{\rightarrow}{+}\overline{X}\overset{Z}{\vert}\overline{ZXY}\cdots\overline{YXZX}$
$\underset{Z}{\uparrow}\overline{YXYZXY}\cdots\overline{YXZ}$	$\overline{XYX}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YXZ}$	$\overset{Z}{\vert}\overline{YXYZXY}\cdots\overline{YX}\overset{\leftarrow}{+}\overline{Z}$	$\overline{X}\overset{\rightarrow}{+}\overline{YX}\overset{Z}{\vert}\overline{ZXY}\cdots\overline{YXZ}$
$\overline{Z}\underset{Z}{\uparrow}\overline{YZYZXY}\cdots\overline{YX}$	$\overline{ZXYX}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YX}$	$\overline{Z}\overset{Z}{\vert}\overline{YZYZXY}\cdots\overline{Y}\overset{\leftarrow}{+}\overline{X}$	$\overline{Z}\overset{\rightarrow}{+}\overline{XYX}\overset{Z}{\vert}\overline{ZXY}\cdots\overline{YX}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\overline{XY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{YXYZ}$	$\overline{XY}\cdots\overline{YXZXYX}\underset{Z}{\uparrow}\overline{Z}$	$\overline{XY}\cdots\overline{YXZ}\overset{Z}{\vert}\overline{YXY}\overset{\leftarrow}{+}\overline{Z}$	$\overline{X}\overset{\rightarrow}{+}\overline{Y}\cdots\overline{YXZXYX}\overset{Z}{\vert}\overline{Z}$
$\overline{ZXY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{YXY}$	$\overline{ZXY}\cdots\overline{YXZXYX}\underset{Z}{\uparrow}$	$\overline{ZXY}\cdots\overline{YXZ}\overset{Z}{\vert}\overline{YX}\overset{\leftarrow}{+}\overline{Y}$	$\overline{Z}\overset{\rightarrow}{+}\overline{XY}\cdots\overline{YXZXYX}\overset{Z}{\vert}$
$\overline{YZXY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{YX}$	$\overline{YZXY}\cdots\overline{YXZXY}\underset{Z}{\uparrow}$	$\overline{YZXY}\cdots\overline{YXZ}\overset{Z}{\vert}\overline{Y}\overset{\leftarrow}{+}\overline{X}$	$\overline{Y}\overset{\rightarrow}{+}\overline{ZXY}\cdots\overline{YXZXY}\overset{Z}{\vert}$
$\overline{XYZXY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{Y}$	$\overline{XYZXY}\cdots\overline{YXZX}\underset{Z}{\uparrow}$		$\overline{X}\overset{\rightarrow}{+}\overline{YZXY}\cdots\overline{YXZX}\overset{Z}{\vert}$

$$\underset{Z}{\uparrow}\overline{YZXY}\cdots\overline{YXZXY}$$
$$\underset{Z}{\uparrow}\overline{XYZXY}\cdots\overline{YXZX}$$
$$\underset{Z}{\uparrow}\overline{YXYZXY}\cdots\overline{YXZ}$$
$$\overline{Z}\underset{Z}{\uparrow}\overline{YZYZXY}\cdots\overline{YX}$$
$$\vdots$$
$$\vdots$$
$$\overline{XY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{YXYZ}$$
$$\overline{ZXY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{YXY}$$
$$\overline{YZXY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{YX}$$
$$\overline{XYZXY}\cdots\overline{YXZ}\underset{Z}{\uparrow}\overline{Y}$$

$$\overline{X}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YXZXY}$$
$$\overline{YX}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YXZX}$$
$$\overline{XYX}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YXZ}$$
$$\overline{ZXYX}\underset{Z}{\uparrow}\overline{ZXY}\cdots\overline{YX}$$
$$\vdots$$
$$\vdots$$
$$\overline{XY}\cdots\overline{YXZXYX}\underset{Z}{\uparrow}\overline{Z}$$
$$\overline{ZXY}\cdots\overline{YXZXYX}\underset{Z}{\uparrow}$$
$$\overline{YZXY}\cdots\overline{YXZXY}\underset{Z}{\uparrow}$$
$$\overline{XYZXY}\cdots\overline{YXZX}\underset{Z}{\uparrow}$$

$$\overset{Z}{|}\overline{YZXY}\cdots\overline{YXZX}\overset{\leftarrow}{+}\overline{Y}$$
$$\overset{Z}{|}\overline{XYZXY}\cdots\overline{YXZ}\overset{\leftarrow}{+}\overline{X}$$
$$\overset{Z}{|}\overline{YXYZXY}\cdots\overline{YX}\overset{\leftarrow}{+}\overline{Z}$$
$$\overline{Z}\overset{Z}{|}\overline{YZYZXY}\cdots\overline{Y}\overset{\leftarrow}{+}\overline{X}$$
$$\vdots$$
$$\vdots$$
$$\overline{XY}\cdots\overline{YXZ}\overset{Z}{|}\overline{YXY}\overset{\leftarrow}{+}\overline{Z}$$
$$\overline{ZXY}\cdots\overline{YXZ}\overset{Z}{|}\overline{YX}\overset{\leftarrow}{+}\overline{Y}$$
$$\overline{YZXY}\cdots\overline{YXZ}\overset{Z}{|}\overline{Y}\overset{\leftarrow}{+}\overline{X}$$

$$\overline{Y}\overset{\rightarrow}{+}\overline{X}\overset{Z}{|}\overline{ZXY}\cdots\overline{YXZX}$$
$$\overline{X}\overset{\rightarrow}{+}\overline{YX}\overset{Z}{|}\overline{ZXY}\cdots\overline{YXZ}$$
$$\overline{Z}\overset{\rightarrow}{+}\overline{XYX}\overset{Z}{|}\overline{ZXY}\cdots\overline{YX}$$
$$\vdots$$
$$\vdots$$
$$\overline{X}\overset{\rightarrow}{+}\overline{Y}\cdots\overline{YXZXYX}\overset{Z}{|}\overline{Z}$$
$$\overline{Z}\overset{\rightarrow}{+}\overline{XY}\cdots\overline{YXZXYX}\overset{Z}{|}$$
$$\overline{Y}\overset{\rightarrow}{+}\overline{ZXY}\cdots\overline{YXZXY}\overset{Z}{|}$$
$$\overline{X}\overset{\rightarrow}{+}\overline{YZXY}\cdots\overline{YXZX}\overset{Z}{|}$$

z magnetic field is moved to the right end!

Final result (k-support operator)

$$h\left(\frac{J_X}{J_Y}-1\right)(k+2)q_{YXZZ\cdots ZYYZ}=0$$

Unless $h=0$ (XYZ model) or $J_X=J_Y$ (XXZ model with a z-magnetic field), this coefficient is zero!

→**Absence of k-support conserved quantity!**
($k \leq L/2$)

Outline

- Background
- Proof (case of 3-support)
- Proof (general case)
- **Extension**

Background structure

The term with largest contiguous support in Hamiltonian is important!

$$H = \sum_i \underbrace{J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z}_{\text{2 sites}} + \underbrace{h S_i^z}_{\text{1 site}}$$

They determine the basic form of possible LCQ.

How the form of LCQ determined?

$$H = \sum_i J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z + h S_i^z$$

Shift products of these three make the form of possible LCQ (i.e., doubling-product).

Unfortunately, deriving inconsistency between above form and terms with smaller support is ad-hoc (model dependent way) at present…

Case of next-nearest Heisenberg chain

$$H = \sum_i J_1 \underbrace{S_i \cdot S_{i+1}}_{\text{2 sites}} + J_2 \underbrace{S_i \cdot S_{i+2}}_{\text{3 sites}}$$

Form of possible k-support LCQ is e.g.,

$$\begin{array}{ccccccccc}
X & \cdot & X & & & & & & \\
 & & Y & \cdot & Y & & & & \\
 & & & & X & \cdot & X & & \\
 & & & & & & Z & \cdot & Z \\
\hline
X & \cdot & Z & \cdot & Z & \cdot & Y & \cdot & Z
\end{array}$$

Case of next-nearest Heisenberg chain

Form of possible k-1 support LCQ is e.g.,

$$
\begin{array}{cccccccc}
X & \cdot & X & & & & & \\
 & & Y & \cdot & Y & & & \\
 & & & & Z & Z & & \\
 & & & & & X & \cdot & X \\
\hline
X & \cdot & Z & \cdot & X & Y & \cdot & X
\end{array}
$$

Considering k-2-support LCQ, we can derive inconsistency.

Future works

- This approach also applies Heisenberg model with staggered magnetic field.
- It is important to clarify general structure.
- Application to S=1 system appears a little difficult.

END