Resource theories with correlated catalyst

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N. Shiraishi and T. Sagawa, Phys. Rev. Lett. 126, 150502 (2021)
R. Takagi and N. Shiraishi, arXiv:2106.12592

Outline

Quantum thermodynamics

- background
- main result (recovery of the second law)

(N. Shiraishi and T. Sagawa, Phys. Rev. Lett. 126, 150502 (2021))

Resource theory of asymmetry

- background
- main result (unbounded convertible power)

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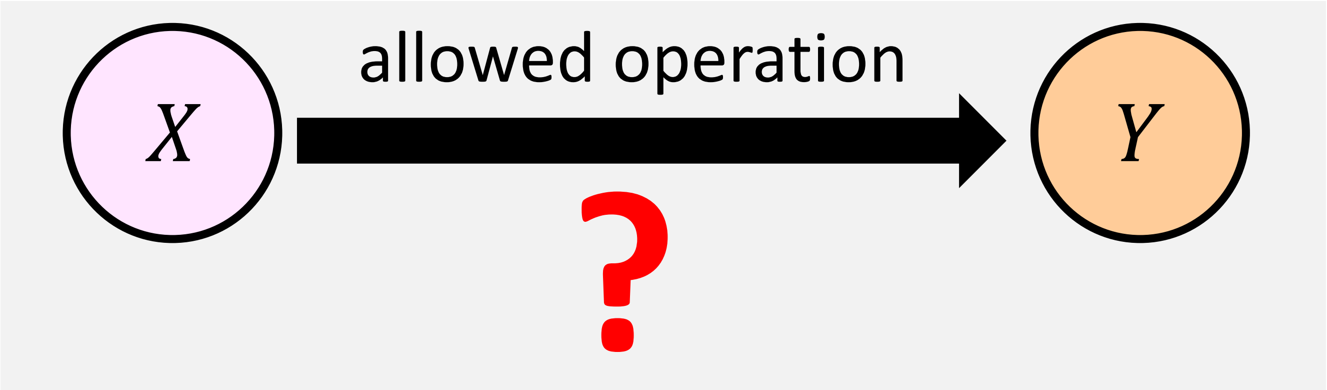
Motivation from macroscopic thermodynamics

X, Y: equilibrium states



In conventional thermodynamics,

(adiabatic operation): $S(X) \leq S(Y)$

(isothermal operation): $F(X) \geq F(Y)$

is the **necessary and sufficient condition**.

(E. H.Lieb and J. Yngvason, Phys. Rep. 310, 1 (1999).)

Situation is completely different in microscopic case

In contrast, in microscopic cases various **new constraints** other than the second law emerge.

Ex) majorization, infinite inequalities…

Questions in quantum thermodynamics

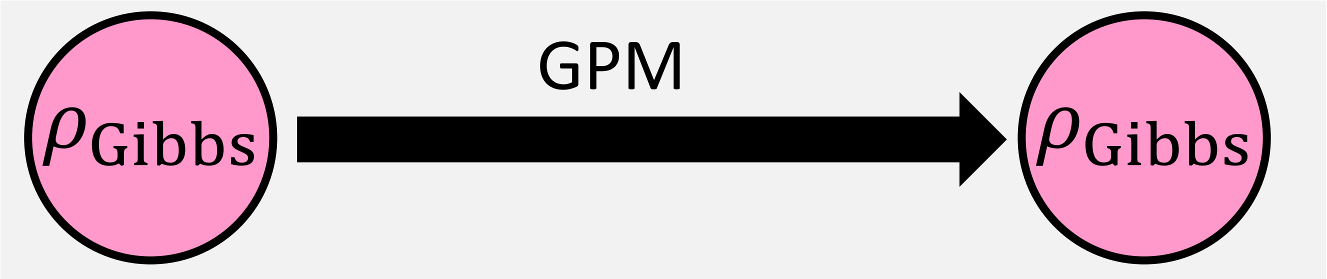
- What is the **necessary and sufficient condition** (nec.&suff.) for state conversions?
- Does **the second law** become a unique criterion? (i.e., a single monotone)

Gibbs-preserving map

We restrict the class of allowed operations to a set of **Gibbs-preserving maps** (with a fixed temperature).

Gibbs-preserving map (GPM)

CPTP map satisfying $\boldsymbol{\Lambda(\rho_{\mathrm{Gibbs}}) = \rho_{Gibbs}}$



GPM is a standard class of thermodynamic operations

Formulation in terms of resource theory

Free operation: $\mathcal{O}$ (Gibbs-preserving map (GPM))

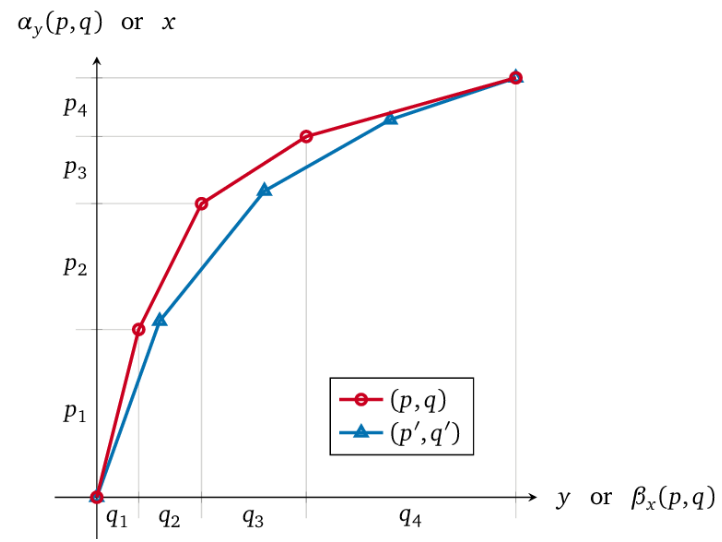
Free state: $\mathcal{F}$ (Gibbs state ρ_{Gibbs})

For any $\Lambda \in \mathcal{O}$ and $\rho \in \mathcal{F}$, $\boldsymbol{\Lambda}(\boldsymbol{\rho}) \in \boldsymbol{\mathcal{F}}$ is satisfied.
(i.e., a free state does not generate a non-free state)

Question:
For given ρ, ρ', what is the nec.&suff. condition for the existence of $\Lambda \in \mathcal{O}$ satisfying $\Lambda(\rho) = \rho'$.

State conversion via GPM (classical)

In the classical case, the nec.&suff. condition for GPM is given by **d-majorization**.



(J. M. Renes, J. Math. Phys. 57, 122202 (2016))

<u>Various proofs</u>:

D. Blackwell, Proc. Math. Statist. and Prob. 93 (1951),

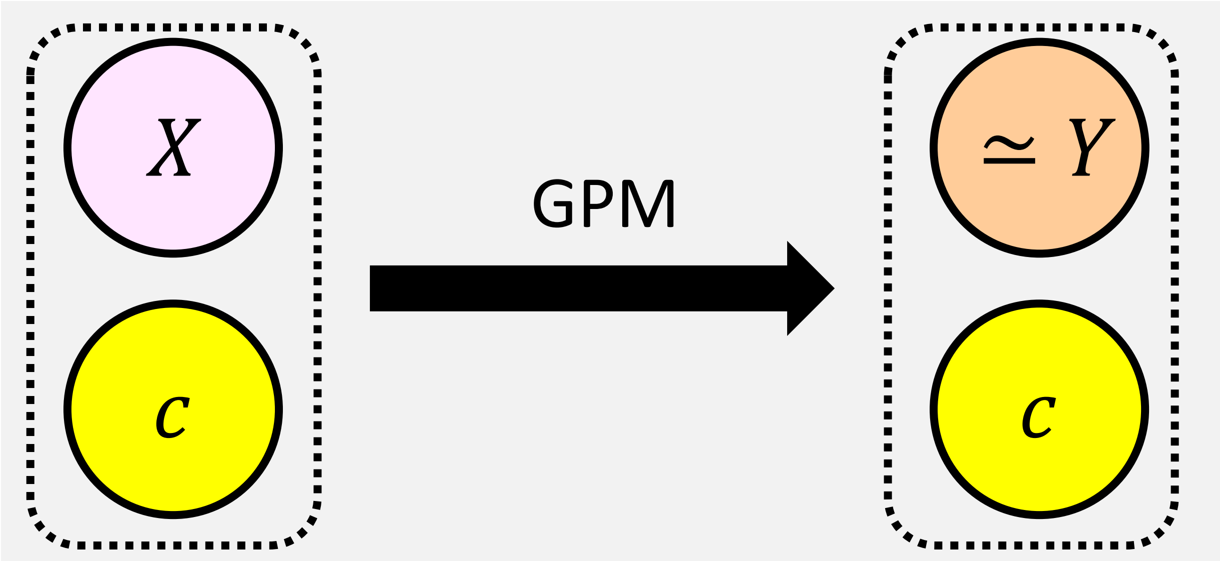
F. Veinott. Jr., Man. Sci. 19, 547 (1971),

E. Ruch, R. Schranner, and T. H. Seligman, J. Chem. Phys. 69, 1 (1978).

N. Shiraishi, J. Phys. A Math. Theor. 53 425301 (2020)

State conversion with catalyst

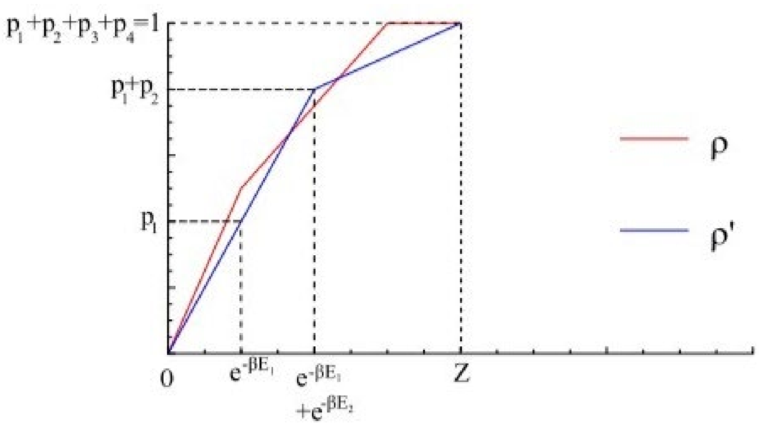
We introduce a **catalyst** system, which does not change through the map but helps conversion.



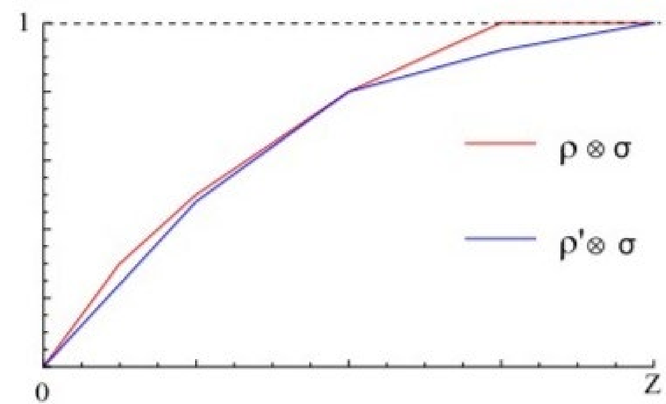
We say that $X \to Y$ is **convertible with a catalyst** if for any $\varepsilon > 0$ there exist proper Λ and c s.t. $\Lambda(X \otimes c) = Y' \otimes c$ with $d(Y, Y') < \varepsilon$

What happens with and without catalyst

Hierarchical relation of Lorenz curves can be changed by catalyst.



without catalyst



with catalyst

(F. Brandao, *et al.*, PNAS 112, 3275 (2015)).

Classical α-Renyi divergence

For two probability distributions p, q, the α-Renyi divergence except $\alpha = 0,1,\infty$ is defined as

$$S_\alpha(p||q) := \frac{\mathrm{sgn}(\alpha)}{\alpha - 1} \ln\left(\sum_i \frac{p_i^\alpha}{q_i^{\alpha-1}}\right)$$

For $\alpha = 0,1,\infty$, we define

$$S_0(p||q) := -\ln\left(\sum_{i;p_i>0} q_i\right)$$

$$S_1(p||q) := \sum_i p_i \ln\frac{p_i}{q_i}$$

$$S_\infty(p||q) := \ln\left(\max_i \frac{p_i}{q_i}\right)$$

State conversion via GPM with catalyst (classical)

<u>Theorem</u>: In classical case, the nec.&suff. condition for GPM with catalyst (with small error ϵ) is given by **infinite inequalities** with α-Renyi divergence:

$$\boldsymbol{F_\alpha(p) \geq F_\alpha(p')} \text{ where } F_\alpha(p) := S_\alpha(p||p_{\mathrm{Gibbs}})$$

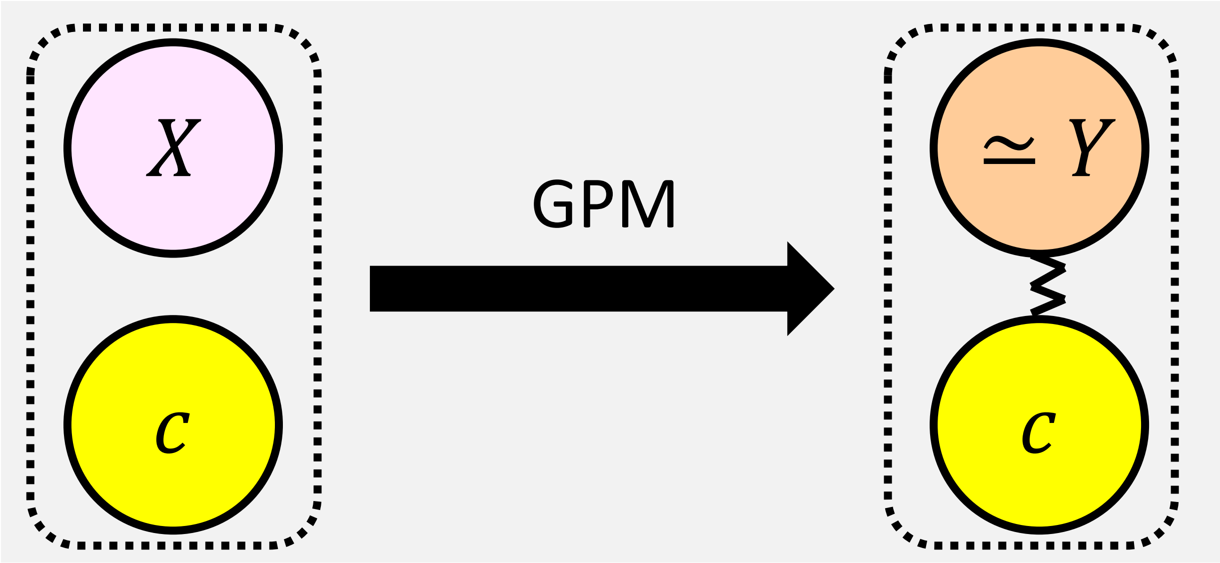
(F. Brandao, *et al.*, PNAS 112, 3275 (2015))

(Key idea: M. Klimesh, arXiv:0709.3680 (2007), S. Turgut, J. Phys. A. 40, 12185 (2007))

(GPM: $\Lambda(p_{\mathrm{Gibbs}} \otimes c_{\mathrm{Gibbs}}) = p_{\mathrm{Gibbs}} \otimes c_{\mathrm{Gibbs}}$)

<u>Proof idea</u>: Constructing an elaborated catalyst state explicitly which satisfies d-majorization condition

State conversion with **correlated** catalyst



$X \to Y$ is **convertible with a correlated catalyst** if for any $\varepsilon > 0$ there exist proper Λ and c s.t. $\Lambda(X \otimes c) = \tau$ with $\mathbf{Tr}_{\boldsymbol{S}}[\boldsymbol{\tau}] = \boldsymbol{c}$, $\boldsymbol{d}(\boldsymbol{Y}, \mathbf{Tr}_{\boldsymbol{C}}[\boldsymbol{\tau}]) < \boldsymbol{\varepsilon}$, and the correlation is arbitrarily small.

State conversion via GPM with catalyst (classical)

<u>Theorem</u>: In classical case, the nec.&suff. condition for GPM with correlated catalyst with small error ϵ is given by **the single free energy with relative entropy (the second law!)**.

$$\boldsymbol{F(p) \geq F(p')} \text{ where } F(p) := S(p||p_{\mathrm{Gibbs}})$$

$$p'_{XY_1} = \begin{pmatrix} \underbrace{\begin{matrix} \delta/n^2 & \cdots & \delta/n^2 \\ \delta/n^2 & \cdots & \delta/n^2 \\ \vdots & & \vdots \\ \delta/n^2 & \cdots & \delta/n^2 \end{matrix}}_{n^2} & \underbrace{\begin{matrix} (p'_1 - 2\delta)/n & \cdots & (p'_1 - 2\delta)/n \\ (p'_2 - 2\delta)/n & \cdots & (p'_2 - 2\delta)/n \\ \vdots & & \vdots \\ (p'_m - 2\delta)/n & \cdots & (p'_m - 2\delta)/n \end{matrix}}_{n} & \begin{matrix} \delta \\ \delta \\ \vdots \\ \delta \end{matrix} \end{pmatrix}$$

<u>proof idea</u>: construct highly elaborated catalyst and reduce to the case of (uncorrelated) catalyst

(M. P. Muller, Phys. Rev. X 8, 041051 (2018).)

Summarizing known results

	Classical	**Quantum**
GPM	d-majorization	**No simple criterion**
GPM with catalyst	Infinite inequalities $F_\alpha(\rho) \geq F_\alpha(\rho')$	**No simple criterion**
GPM with correlated catalyst	The second law $F(\rho) \geq F(\rho')$	**???**

There is a conjecture

In the quantum case, it is conjectured that…

<u>Conjecture</u>

In the quantum case, the nec.&suff. condition to convert $\rho \to \rho'$ by GPM with a correlated catalyst is the second law with quantum relative entropy ($F(\rho) = S(\rho || \rho_{\mathrm{Gibbs}})$):

$$F(\rho) \geq F(\rho')$$

(H. Wilming, R. Gallego, and J. Eisert, Entropy 19, 241 (2017).
M. Lostaglio and M. P. Muller, Phys. Rev. Lett. 123, 020403 (2019))

Single thermodynamic potential!

We need new approach!

	Classical	Quantum
GPM	d-majorization	**No simple criterion**
GPM with catalyst	Infinite inequalities $F_\alpha(\rho) \geq F_\alpha(\rho')$	**No simple criterion**
GPM with correlated catalyst	The second law $F(\rho) \geq F(\rho')$	**???**

used in proof

used in proof

We cannot follow the approach in classical cases.

Completely new approach is needed!

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- **main result (recovery of the second law)**

(N. Shiraishi and T. Sagawa, Phys. Rev. Lett. 126, 150502 (2021))

Resource theory of asymmetry

- background
- main result (unbounded convertible power)

(R. Takagi and N. Shiraishi, arXiv:2106.12592)

Main result

Main result

In the quantum case, nec.&suff. condition to convert $\rho \to \rho'$ by GPM with a correlated catalyst is the second law with quantum relative entropy ($F(\rho) = S(\rho||\rho_{\mathrm{Gibbs}})$):

$$F(\rho) \geq F(\rho')$$

(N. Shiraishi and T. Sagawa, Phys. Rev. Lett. 126, 150502 (2021))

The conjecture is solved in positive.

The second law is recovered in quantum microscopic systems.

Necessary part (easy part)

Let $\Lambda(\rho \otimes c) = \tau$ with $\mathrm{Tr}_S[\tau] = c$.
Using the additivity, superadditivity, and monotonicity of relative entropy, we have

$$\begin{aligned} & S(\rho||\rho_{\mathrm{Gibbs}}) + S(c||c_{\mathrm{Gibbs}}) \\ = & S(\rho \otimes c||\rho_{\mathrm{Gibbs}} \otimes c_{\mathrm{Gibbs}}) \\ \geq & S(\tau||\rho_{\mathrm{Gibbs}} \otimes c_{\mathrm{Gibbs}}) \\ \geq & S(\underline{\mathrm{Tr}_C[\tau]}||\rho_{\mathrm{Gibbs}}) + S(c||c_{\mathrm{Gibbs}}) \end{aligned}$$

arbitrarily close to ρ'

Three-step proof strategy of the sufficient part

1: Sufficient condition for state conversions (measurement-preparation method)

2: Nec. and suff. Condition for asymptotic conversions

3: Reduction from asymptotic conversions to (correlated-)catalytic conversions

(Note: This proof technique is not only for q-thermo but for other resource theories)

Before step 1: quantum α-Renyi divergence

Renyi-0 divergence: $S_0(\rho||\sigma) := -\ln[\mathrm{Tr}[P_\rho \sigma]]$

(P_ρ: projection onto the support of ρ)

Renyi-∞ divergence: $S_\infty(\rho||\sigma) := \ln[\min\{\lambda|\rho \leq \lambda\sigma\}]$

$S_\infty(\rho||\sigma)$ (highest distinguishability)

VI

$S_1(\rho||\sigma)$ (KL divergence (relative entropy))

VI

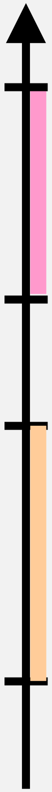
$S_0(\rho||\sigma)$ (lowest distinguishability)

Step 1: sufficient condition for state conversion

<u>Theorem 1</u>: There exists a GPM with $\Lambda(\rho) = \rho'$ if

$$\boldsymbol{S_0(\rho||\rho_{\mathrm{Gibbs}}) \geq S_\infty(\rho'||\rho_{Gibbs})}$$

(P. Faist and R. Renner, Phys. Rev. X 8, 021011 (2018).)



<u>Intuitive picture</u>

ρ is more distinguishable than ρ' from ρ_{Gibbs} in any sense.

→ ρ is convertible to ρ' via GPM

Proof of Theorem 1: Measurement-preparation method

i. Perform a measurement with $\{P_\rho, 1 - P_\rho\}$.

$$\rho \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \rho_{\mathrm{Gibbs}} \to \begin{pmatrix} k \\ 1-k \end{pmatrix} \text{ with } k = e^{-S_0(\rho || \rho_{\mathrm{Gibbs}})}$$

ii. Prepare a quantum state as $\begin{pmatrix} p \\ q \end{pmatrix} \to p\rho' + q\eta$ with

$$\eta = \frac{\rho_{\mathrm{Gibbs}} - k\rho'}{1-k}$$

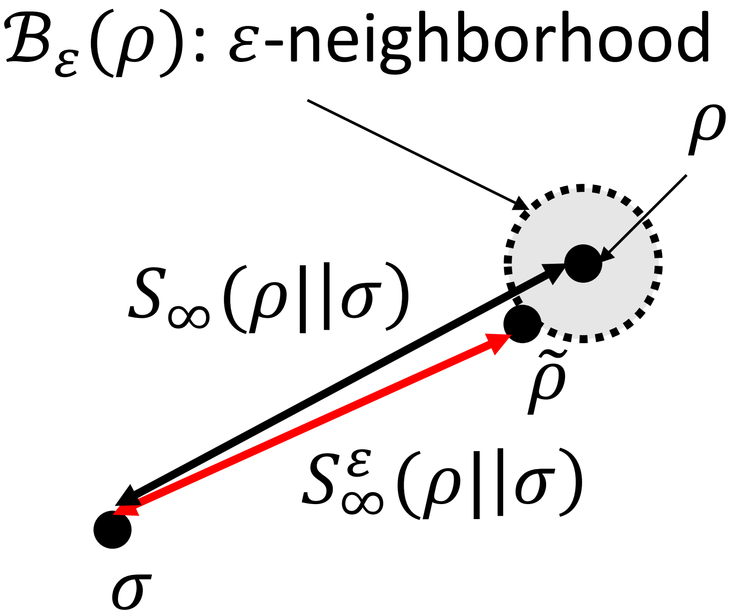
($S_0(\rho||\rho_{\mathrm{Gibbs}}) \geq S_\infty(\rho'||\rho_{Gibbs})$ confirms that η is positive-semidefinite)

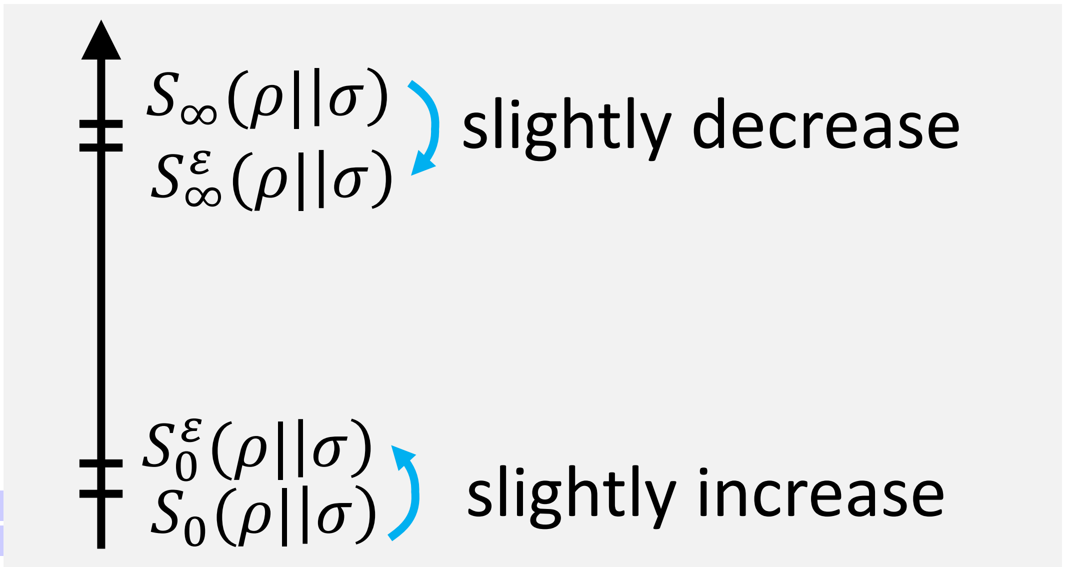
Before step 2: ε-smoothing

Def: smoothed divergence

$$S_\infty^\varepsilon(\rho||\sigma) = \min_{\tilde{\rho}\in\mathcal{B}_\varepsilon(\rho)} S_\infty(\tilde{\rho}||\sigma)$$

$$S_0^\varepsilon(\rho||\sigma) = \max_{\tilde{\rho}\in\mathcal{B}_\varepsilon(\rho)} S_0(\tilde{\rho}||\sigma)$$





(Note: previous sufficient condition is still valid for ε-smoothed version)

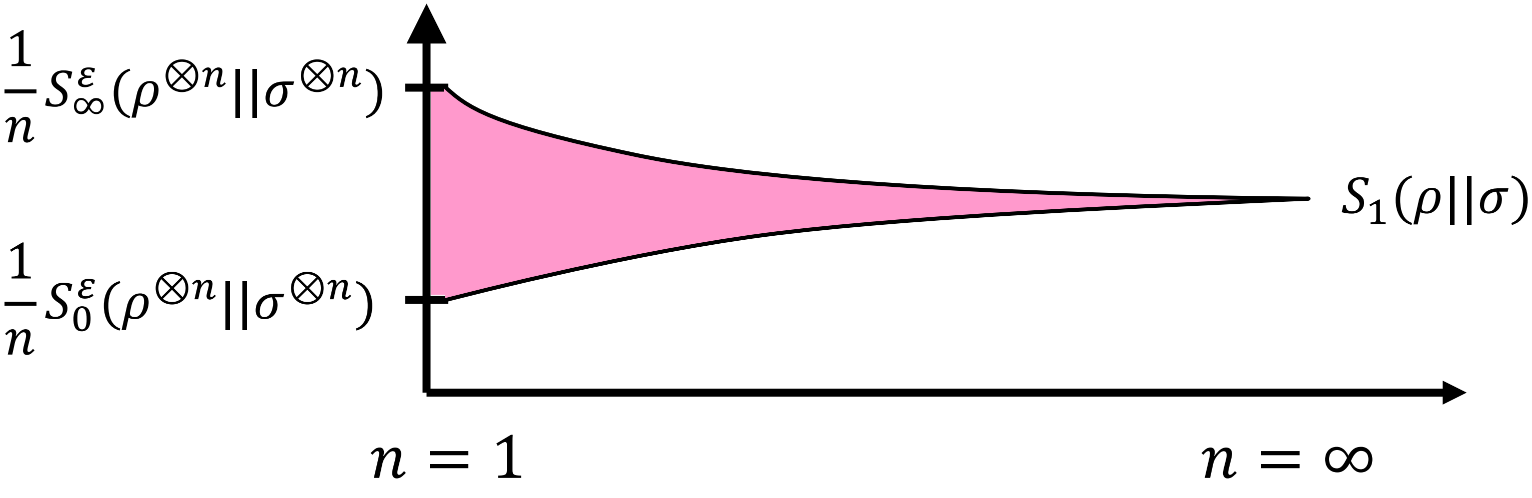
Step 2: Convergence of α-Renyi divergence

Using the quantum Stain's lemma, we have

Theorem 2: For any $0 < \varepsilon < 1/2$ and $0 \leq \alpha \leq \infty$,

$$\lim_{n\to\infty} \frac{1}{n} S_\alpha^\varepsilon(\rho^{\otimes n}||\sigma^{\otimes n}) = S_1(\rho||\sigma)$$

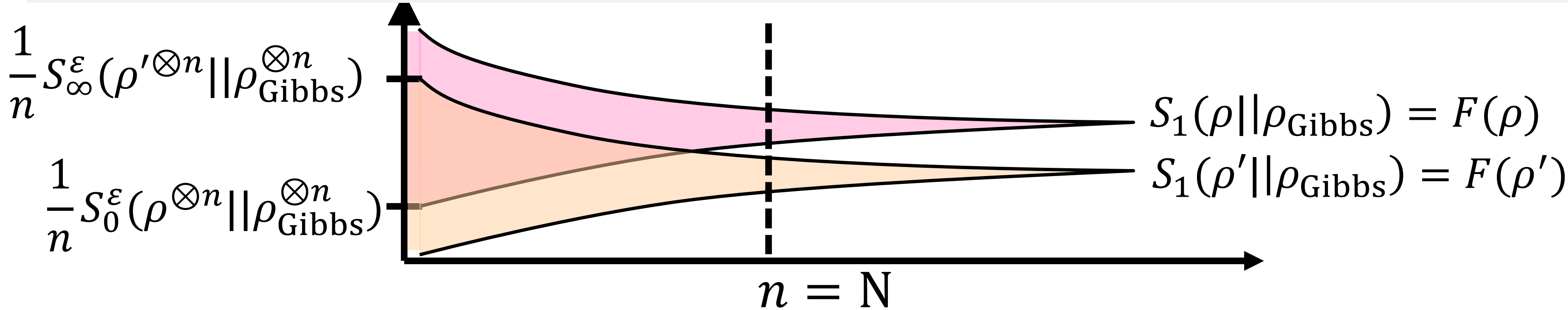
(N. Datta, IEEE Trans. 55, 2816 (2009))



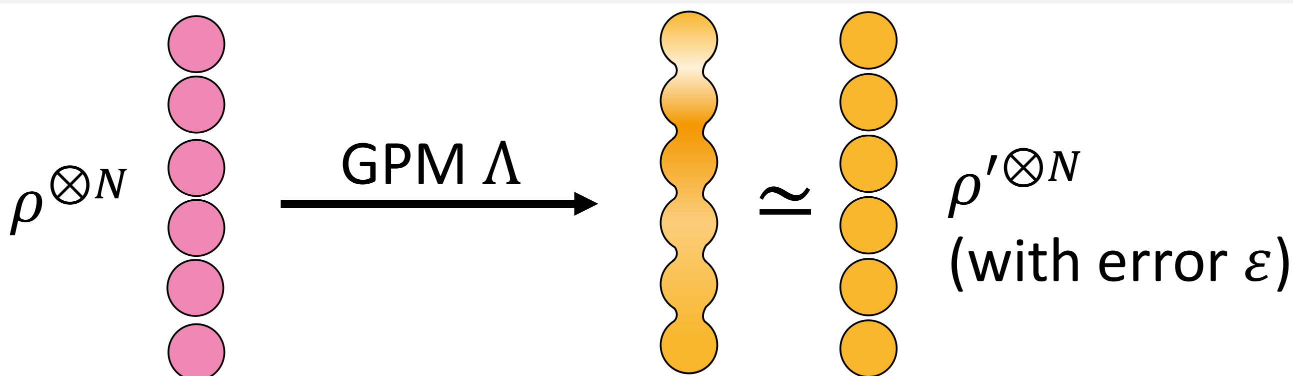
Combining Theorem 1 and 2

From Theorem 2, if $F(\rho) \geq F(\rho')$, there exists N satisfying

$$\frac{1}{N} S_{\infty}^{\varepsilon}(\rho'^{\otimes N} || \rho_{Gibbs}^{\otimes N}) < \frac{1}{N} S_{0}^{\varepsilon}(\rho^{\otimes N} || \rho_{Gibbs}^{\otimes N})$$



From Theorem 1, there exists a GPM Λ with $\boldsymbol{\Lambda(\rho^{\otimes N}) \simeq \rho'^{\otimes N}}$

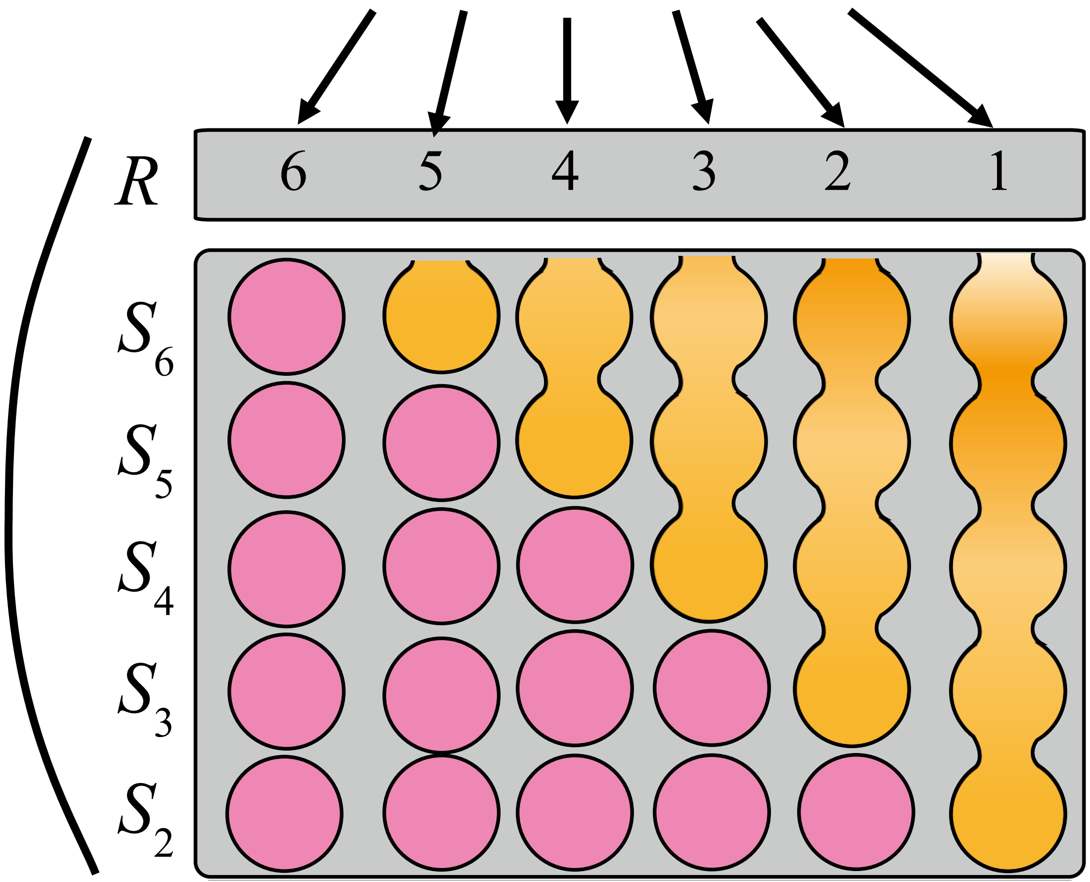


Step 3: From asymptotic to catalytic

classical mixture of 6 states with equal weights

Ex) $N = 6$

catalyst



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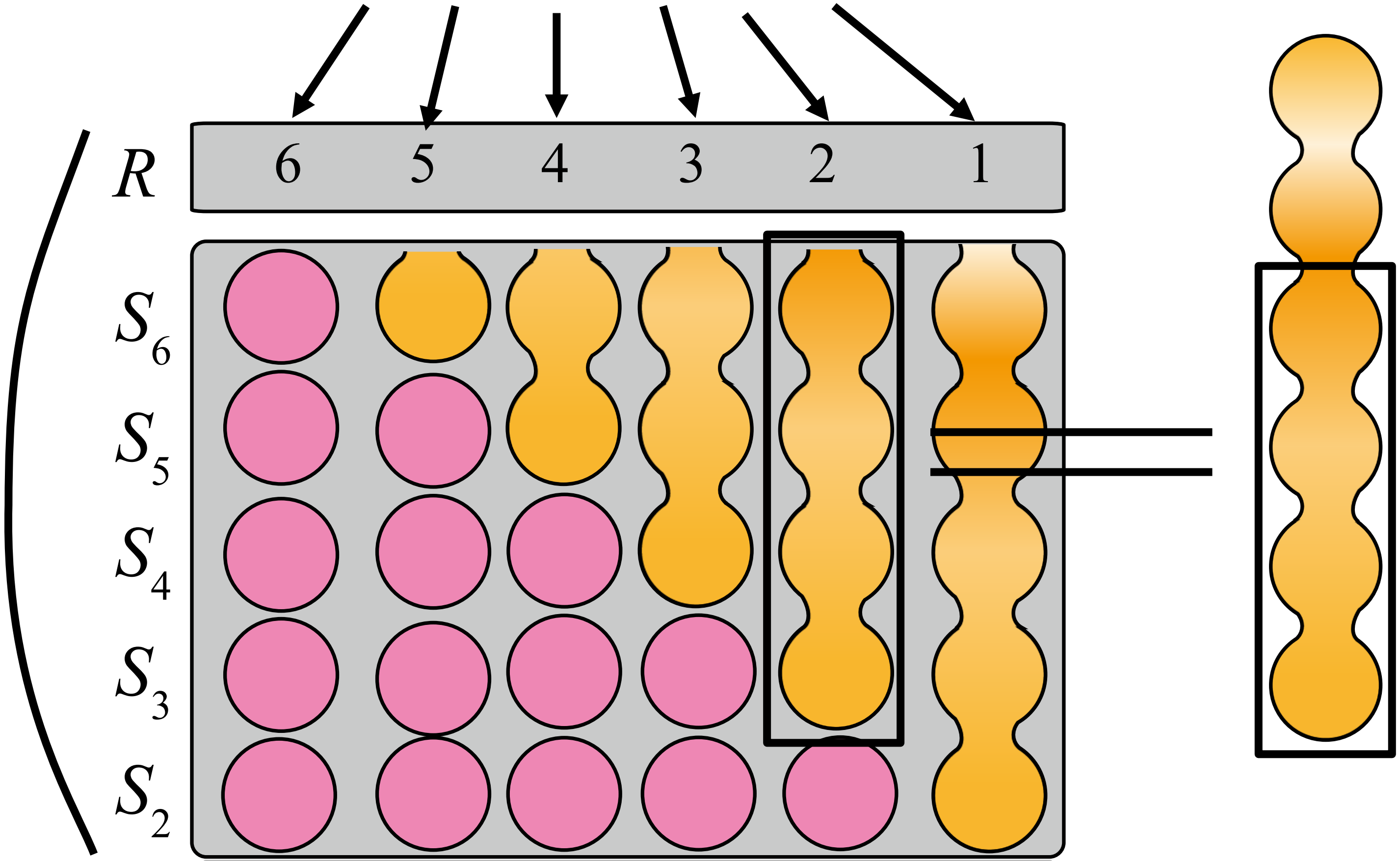


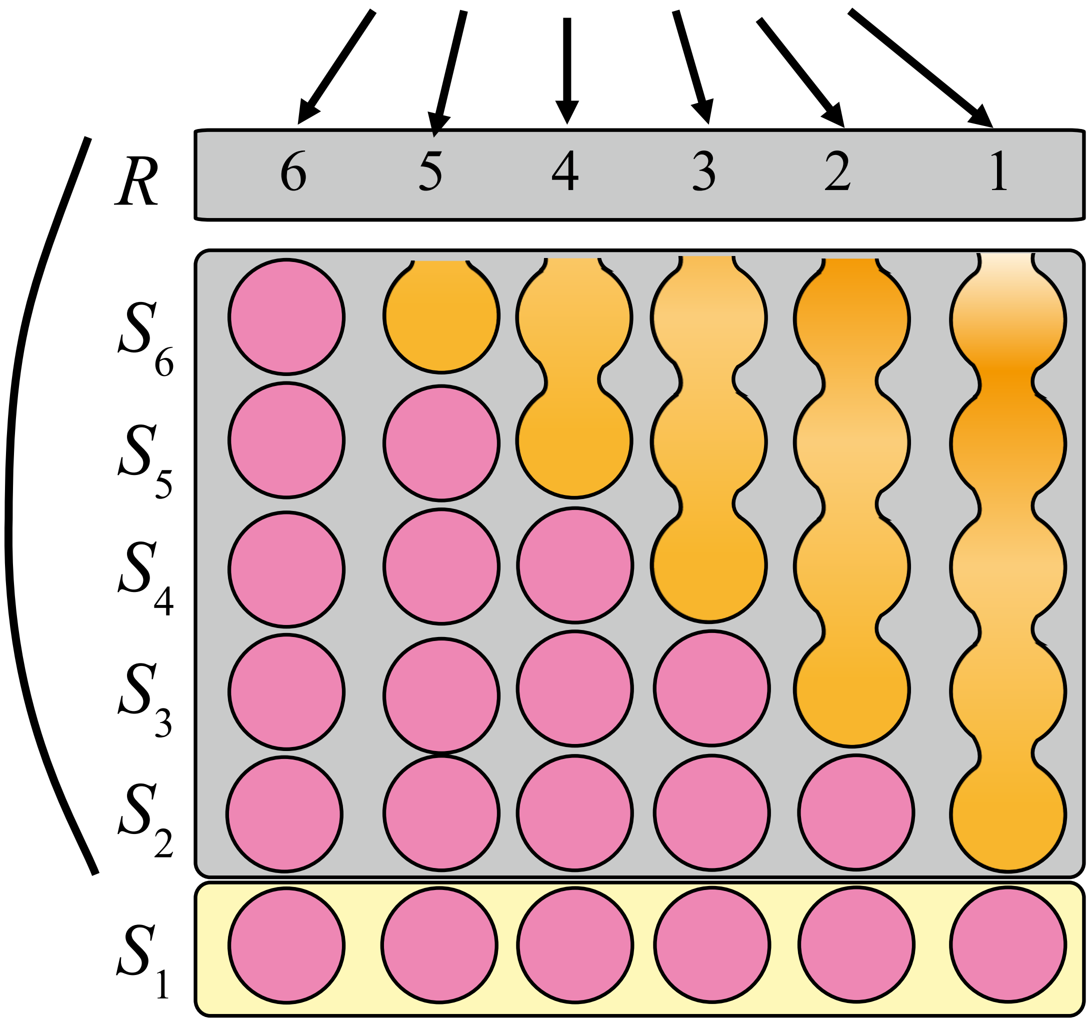
Step 3: From asymptotic to catalytic

classical mixture of 6 states with equal weights

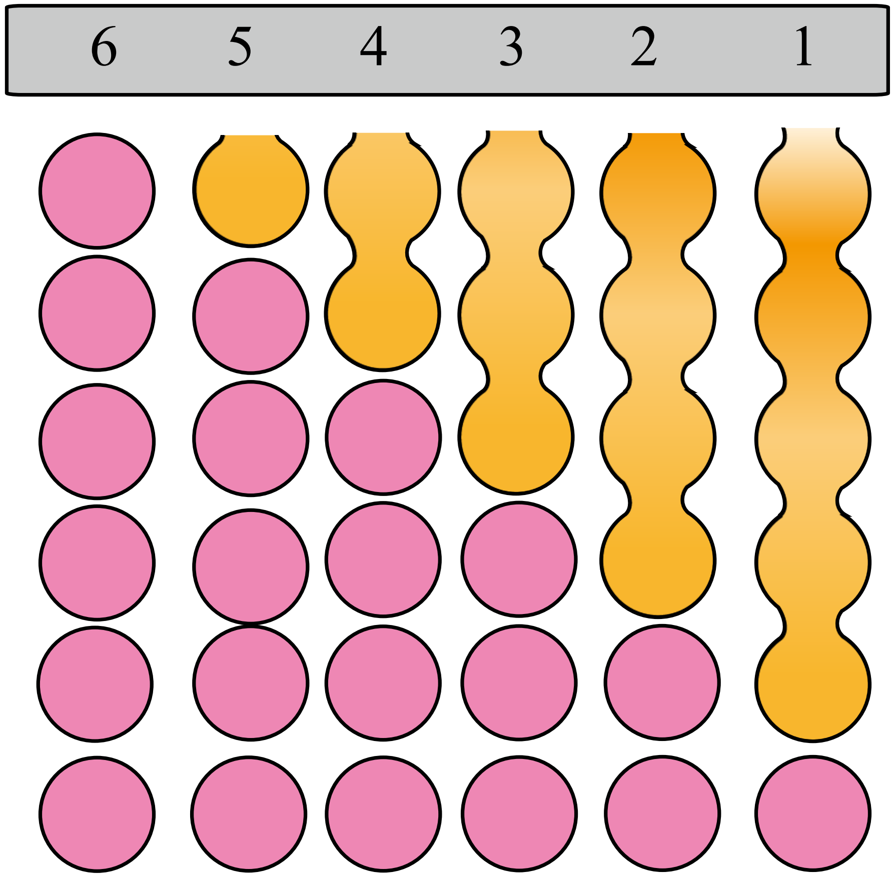
Ex) $N = 6$

catalyst

system



Step 3: From multi-copy to catalytic



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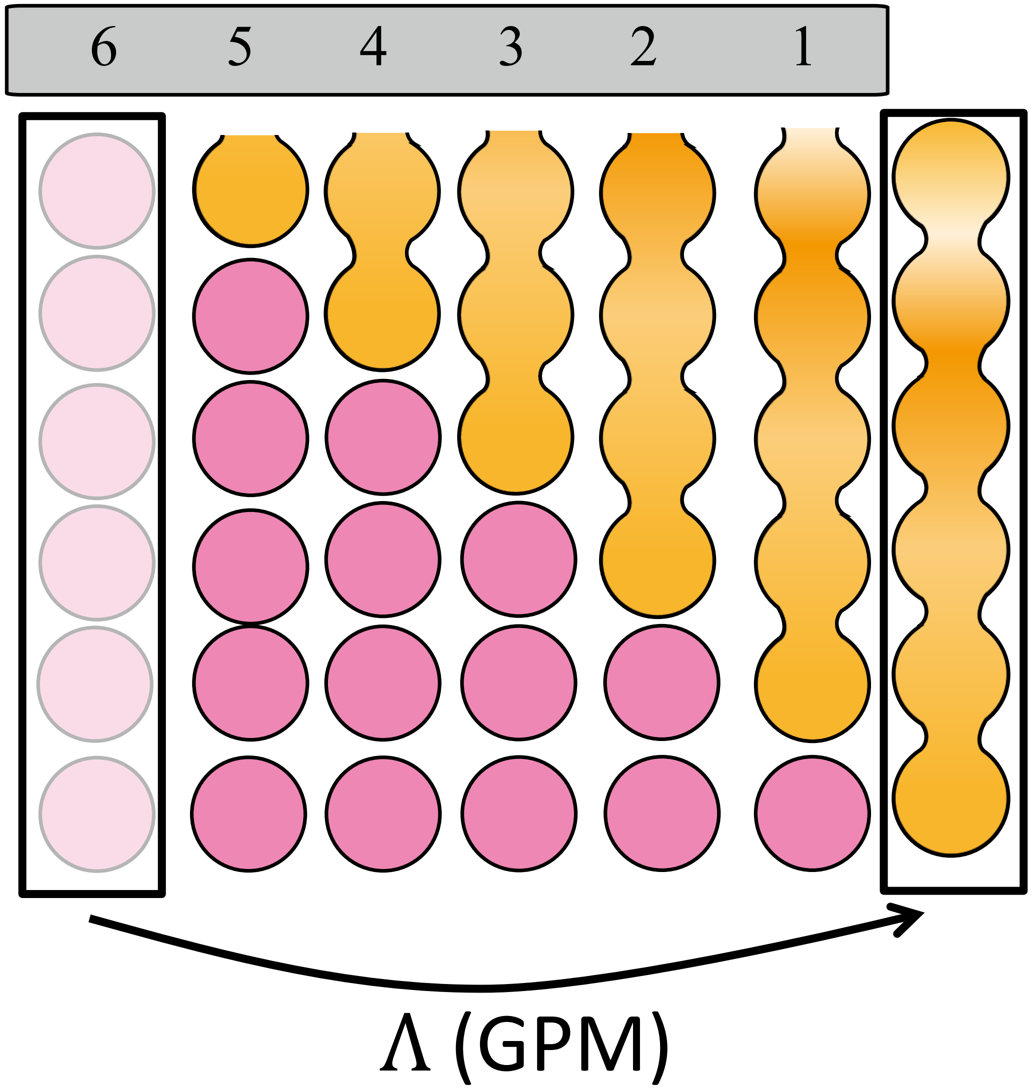
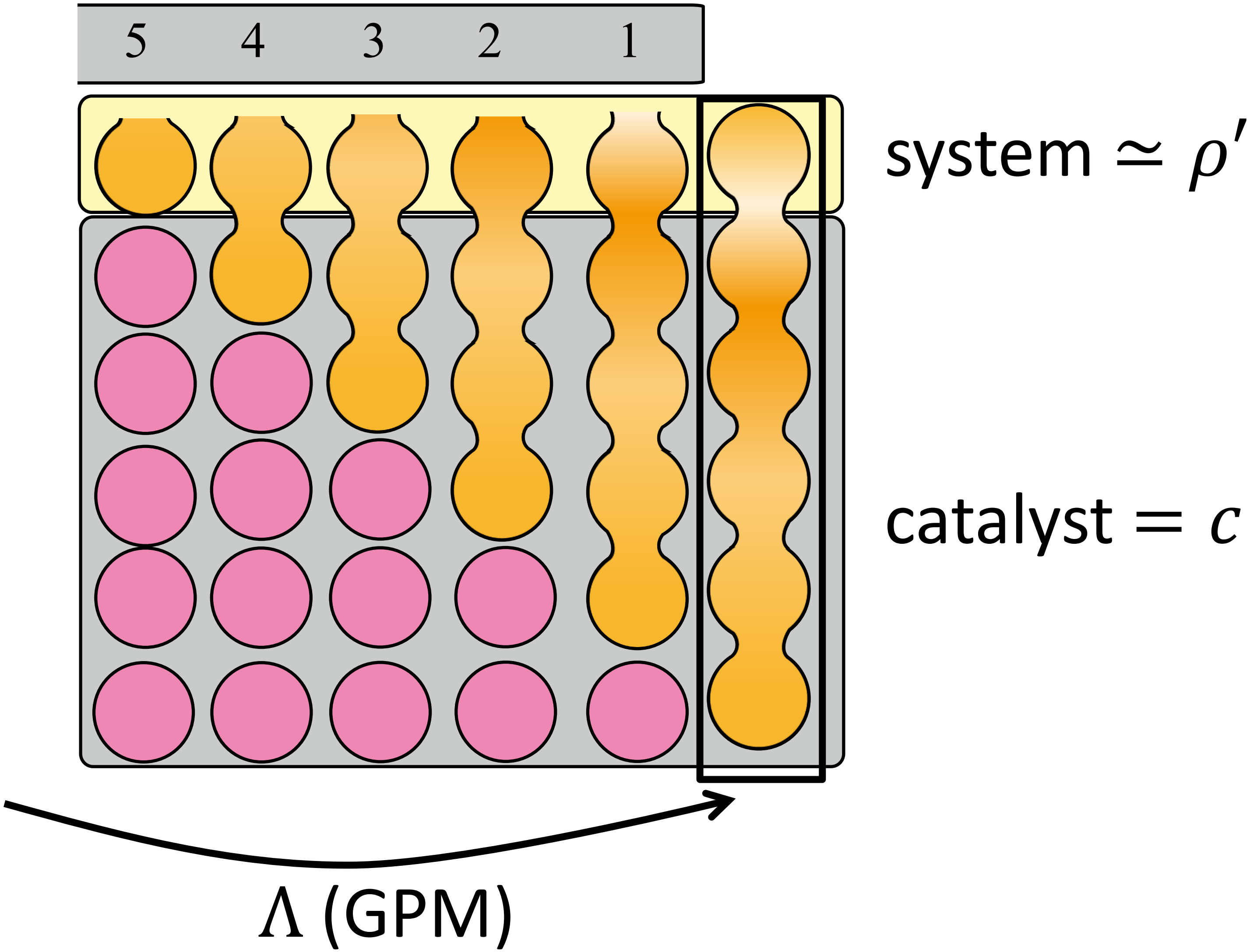
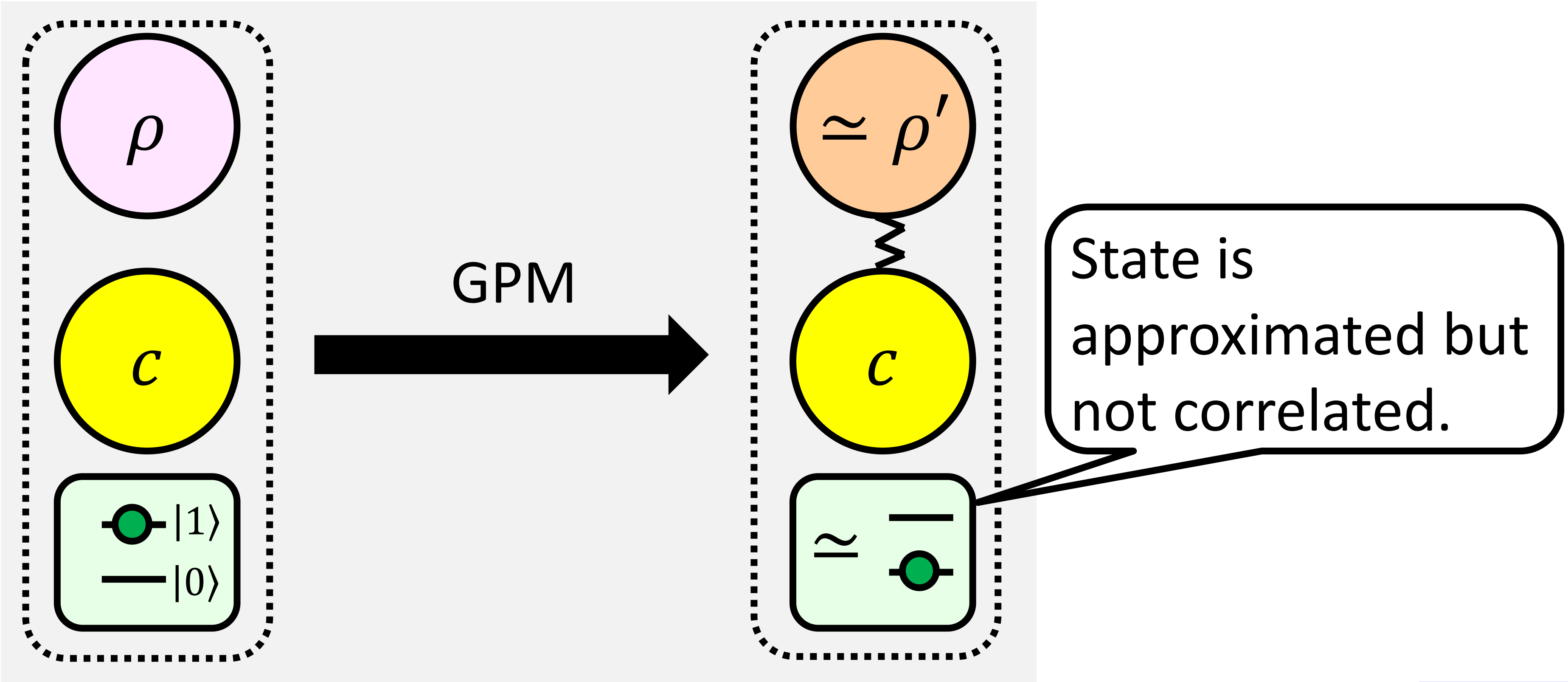


Step 3: From multi-copy to catalytic



Remark 1: case with work storage

We introduce a two-level system called **work storage**, which compensate the energy change.



Conversion against free energy difference with the aid of work storage

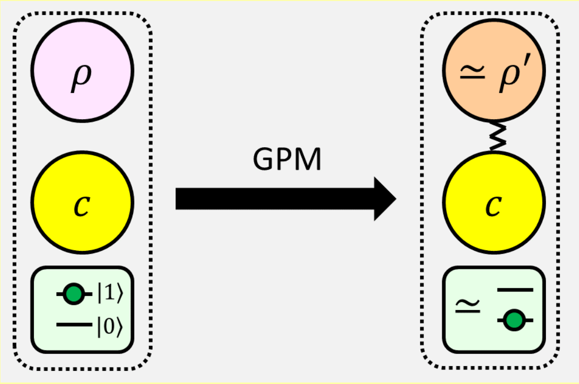
Theorem

If $F(\rho) < F(\rho')$, for any $\varepsilon > 0$, there exist a catalyst, a work storage with $\boldsymbol{\Delta E = F(\rho') - F(\rho)}$, and a GPM Λ such that

$$\boldsymbol{\Lambda(\rho \otimes c \otimes |1\rangle\langle 1|) = \tau \otimes \omega}$$

with

$$d(\mathrm{Tr}_C[\tau], \rho') < \varepsilon$$
$$\mathrm{Tr}_S[\tau] = c$$
$$d(\omega, |0\rangle\langle 0|) < \varepsilon$$



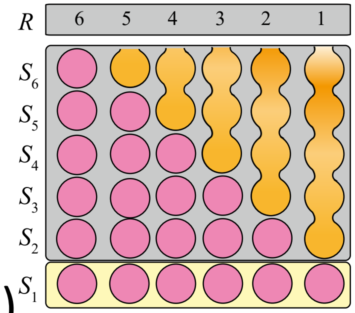
(N. Shiraishi and T. Sagawa, Phys. Rev. Lett. 126, 150502 (2021))

(Remark: the case of work extraction is not shown at present)

Remark 2: Applications to other resource theories

This proof method applies other resource theories.

Sufficient condition for **asymptotic conversion** is also that for **catalytic conversion** (see Step 3).



<u>Various applications of our proof method</u>:

Entropy conjecture: (H. Wilming, Phys. Rev. Lett. 127, 260402 (2021))
Entanglement: (T. V. Kondra, *et al.*, Phys. Rev. Lett. 127, 150503 (2021))
Teleportation: (P. Lipka-Bartosik and P. Skrzypczyk, PRL 127, 080502 (2021))

See N. Shiraishi and T. Sagawa, Phys. Rev. Lett. 126, 150502 (2021) and R. Takagi and N. Shiraishi, arXiv:2106.12592 for further discussion.

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Resource theory of asymmetry

Free operation: energy conserving operation
Free state: incoherent state w.r.t. energy eigenstates

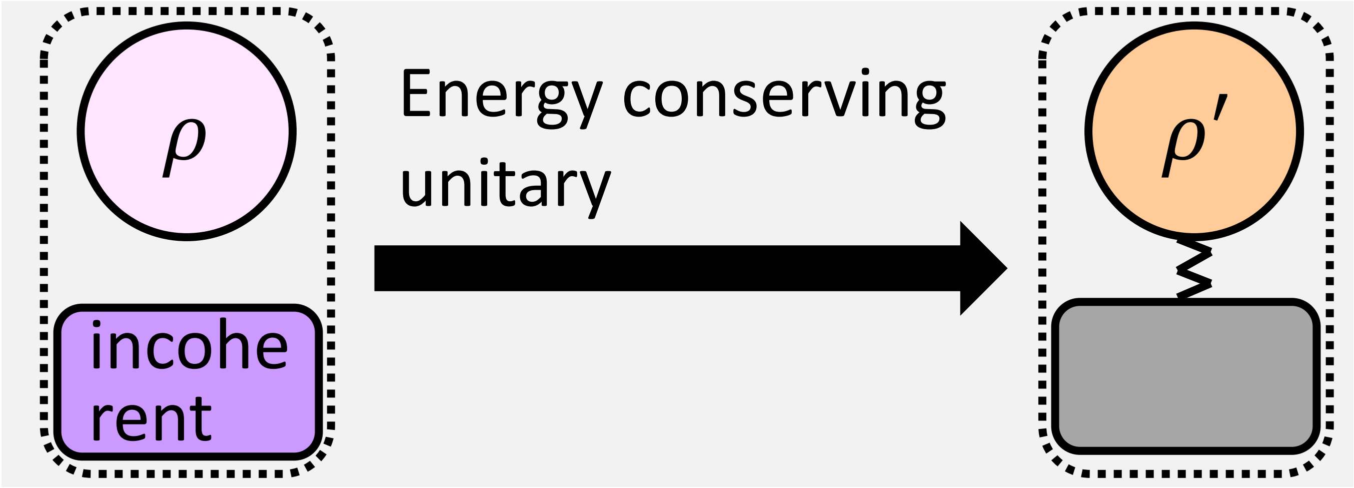
symmetric = incoherent w.r.t. energy
asymmetric = coherent w.r.t. energy

We examine whether $\Lambda: \rho \to \rho'$ is possible with a symmetric map Λ (energy-conserving map).

(Note: we consider U(1) symmetry in this talk)

Physical motivation

All physical processes conserves energy
→Symmetry with energy basis naturally appears!

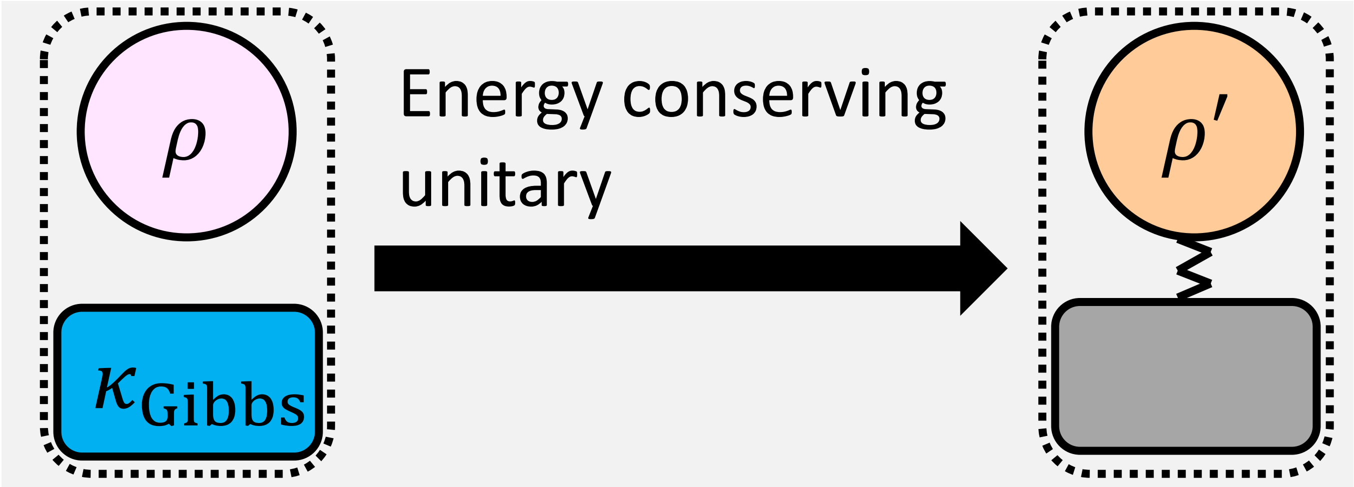


High coherence between energy eigenstates
can be a resource.

Motivation from q-thermo

Def: $\rho \to \rho'$ is convertible via **Thermal operation (TO)** if there exist an energy conserving unitary and an auxiliary system A with a state κ_{Gibbs} such that

$$\mathrm{Tr}_{\mathrm{A}}\left[U(\rho \otimes \kappa_{\mathrm{Gibbs}})U^{\dagger}\right] = \rho'$$



(Note: κ_{Gibbs} is an incoherent state)

Status of thermal operation

Classical case: GPM=TO

(M. Horodecki and J. Oppenheim, Nat. Comm. 4, 2059 (2013), N. Shiraishi, J. Phys. A Math. Theor. 53 425301 (2020))

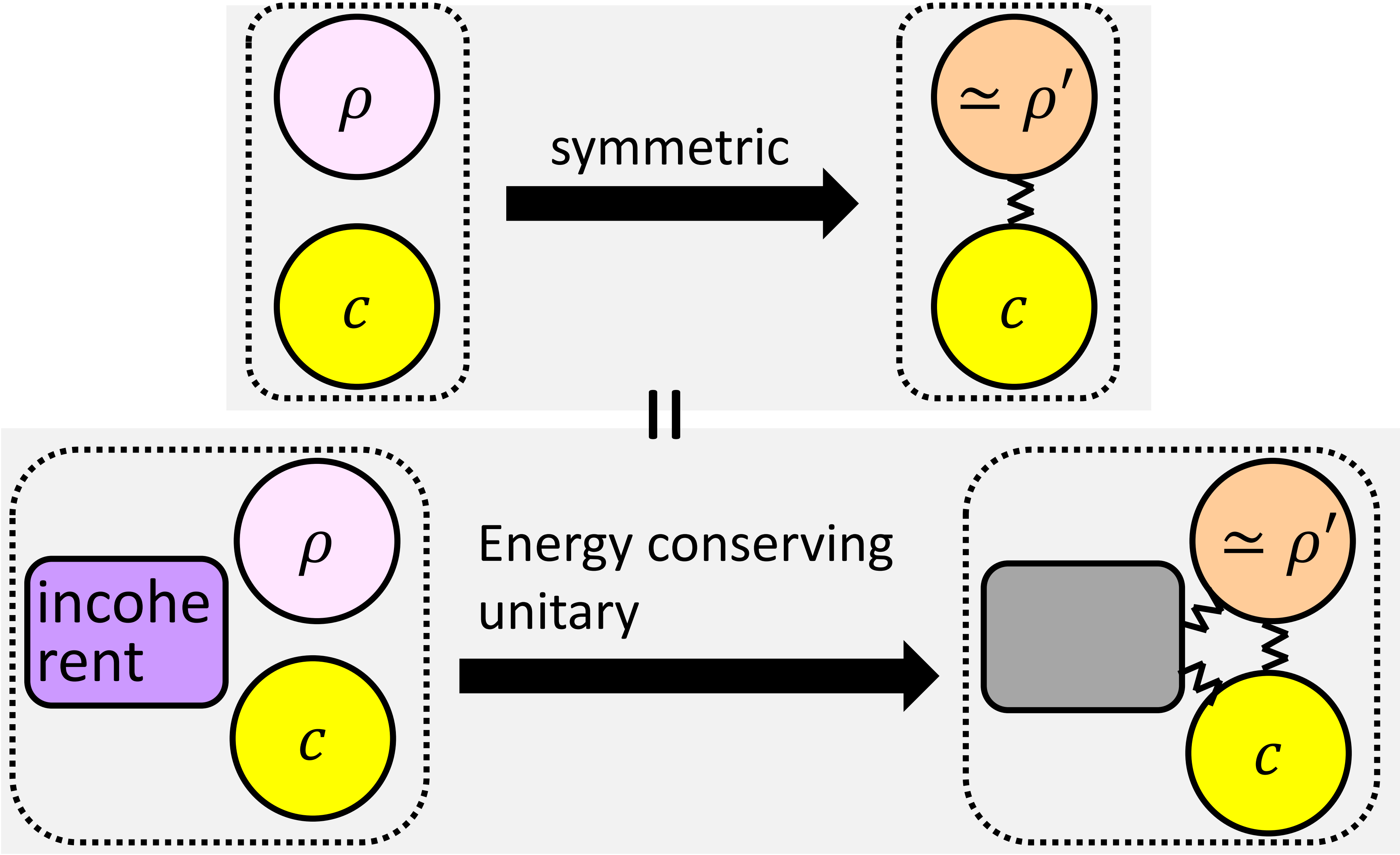
Quantum case: GPM⊃TO and GPM≠TO

(P. Faist, J. Oppenheim, and R. Renner, New J. Phys. 17, 043003 (2015))

Symmetric map (TO) cannot create energy coherence.

$$|E_1\rangle \not\longrightarrow \frac{1}{\sqrt{2}}(|E_1\rangle + |E_2\rangle)$$

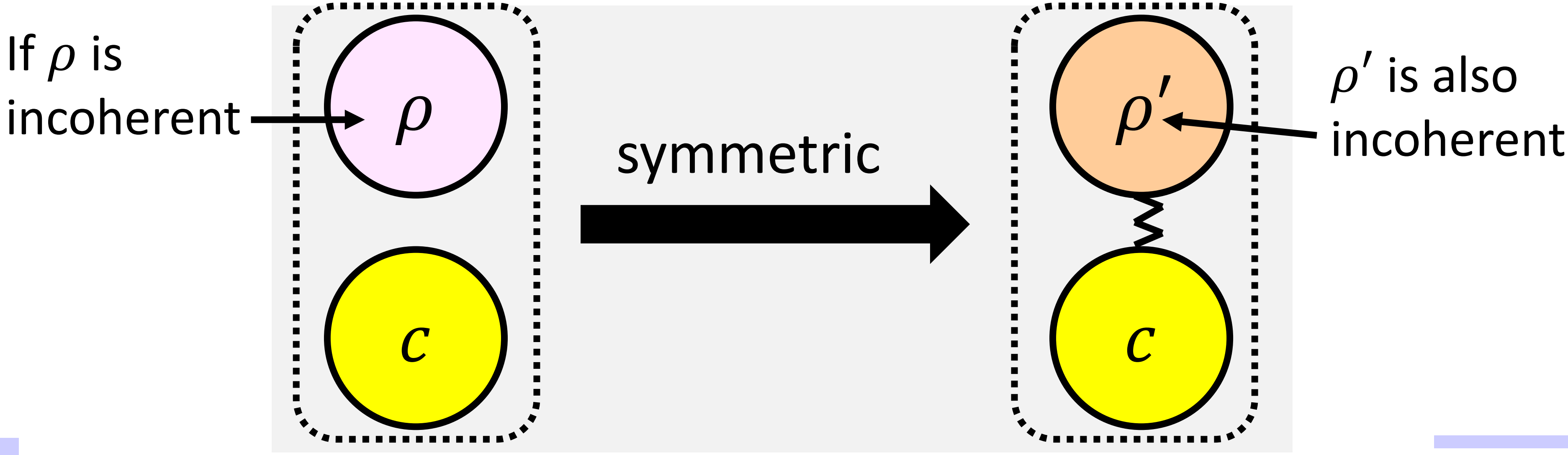
Symmetric map with correlated catalyst



No broadcasting theorem

<u>Theorem</u>: Let ρ be an incoherent state, and Λ be a symmetric map. If $\mathrm{Tr}_S[\Lambda(\rho \otimes c)] = c$, then $\rho' = \mathrm{Tr}_C[\Lambda(\rho \otimes c)]$ is an incoherent state.

(M. Lostaglio and M. P. Muller, Phys. Rev. Lett. 123, 020403 (2019),
I. Marvian and R. W. Spekkens, Phys. Rev. Lett. 123, 020404 (2019).)



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(M. Lostaglio and M. P. Muller, Phys. Rev. Lett. 123, 020403 (2019),
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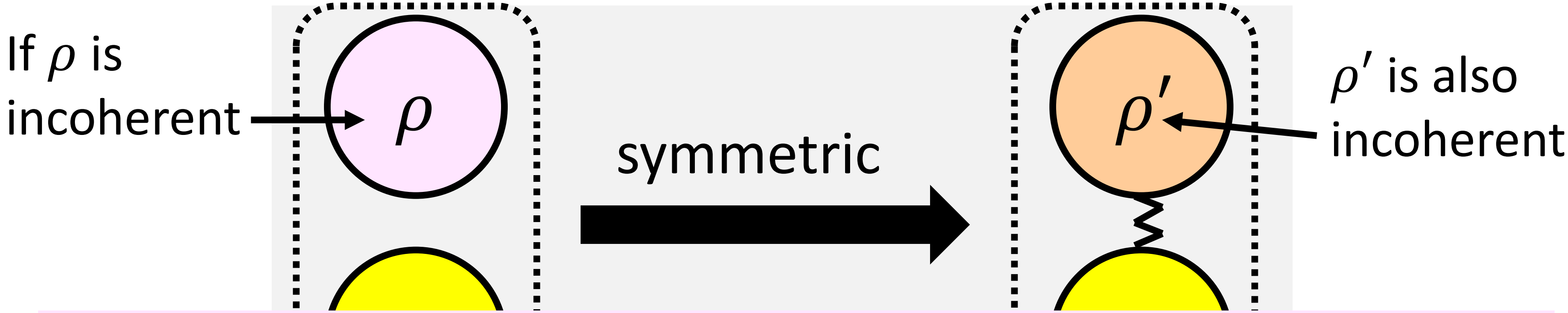

In this case, a catalyst gives no advantage to symmetric maps.

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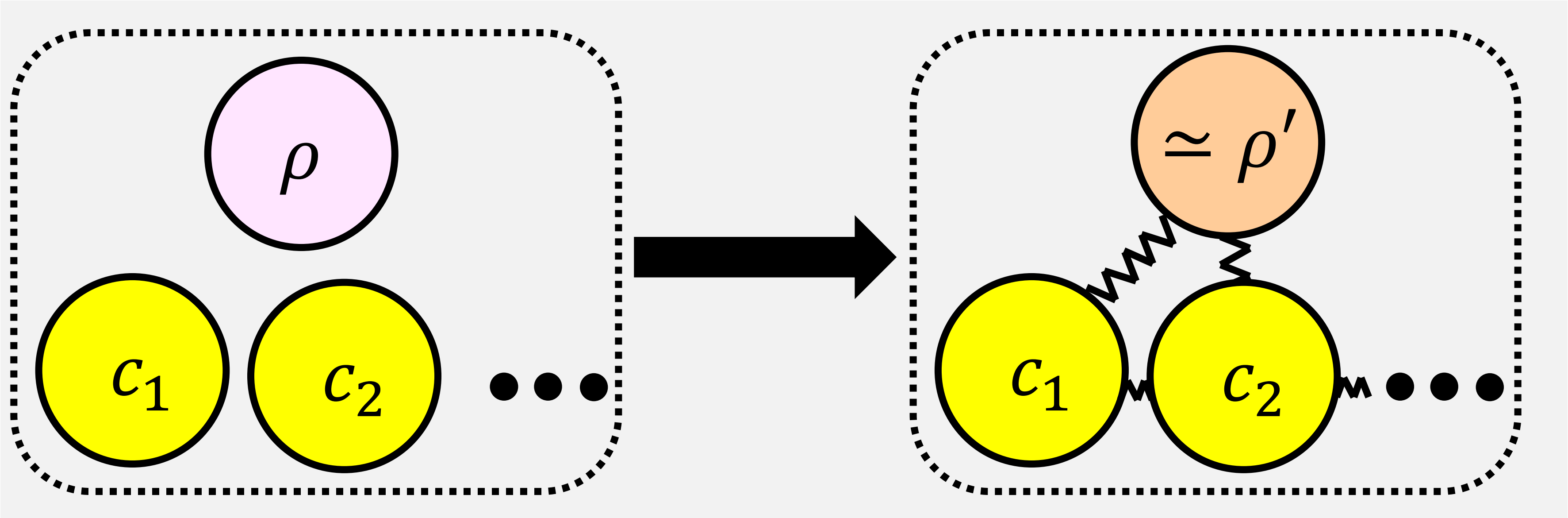
(R. Takagi and N. Shiraishi, arXiv:2106.12592)

Marginal catalytic operation

Suppose that catalysts can correlate with each other (marginal catalytic operation).



We consider the power of symmetric maps with marginal catalysts.

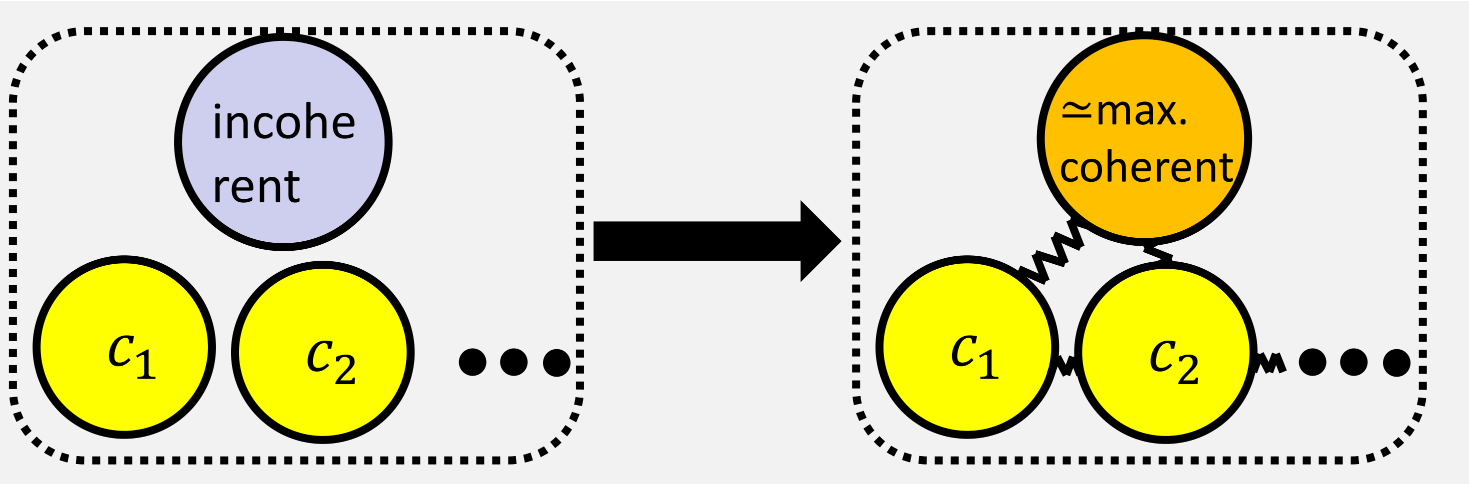
Symmetric map with marginal catalyst is trivial.

<u>Theorem</u>: Any ρ can be converted to any ρ' via a symmetric map with marginal catalysts.

(R. Takagi and N. Shiraishi, arXiv:2106.12592)

Coherence gives no restriction!

(i.e, the resource theory of asymmetry (unspeakable coherence) with marginal catalyst is trivial.)



Remark: Most resource theories with marginal catalyst is not trivial

Theorem: If there exists a resource measure $R(\rho)$ satisfying superadditivity and tensor-product additivity, then conversion $\rho \to \rho'$ with marginal catalysts implies $R(\rho) \geq R(\rho')$.

(R. Takagi and N. Shiraishi, arXiv:2106.12592)

Most resource theories (entanglement, thermodynamics, (speakable) coherence…) have such a measure, and hence do not become trivial even with marginal catalysts.

Key protocol for triviality

Let $\Sigma(\eta) := \begin{pmatrix} 1/2 & \eta/2 \\ \eta/2 & 1/2 \end{pmatrix}$

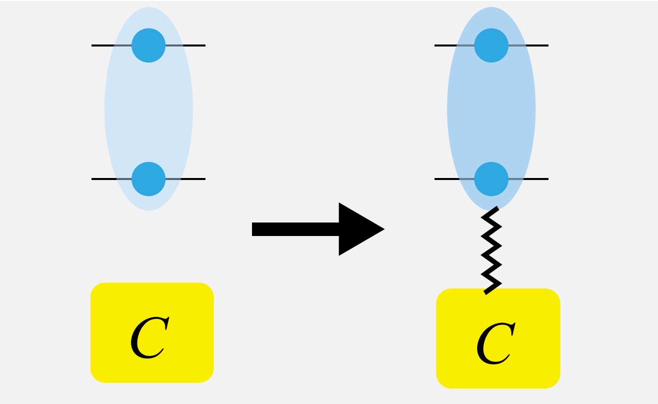
$\eta = 0$: incoherent
$\eta = 1$: maximally coherent

For $0 < \eta < 1$, there exist a two-state catalyst $\Gamma(\eta)$ and a symmetric map Λ such that

$$\mathrm{Tr}_C\left[\Lambda\left(\Sigma(\eta) \otimes \Gamma(\eta)\right)\right] = \Sigma(\eta'),$$
$$\mathrm{Tr}_S\left[\Lambda\left(\Sigma(\eta) \otimes \Gamma(\eta)\right)\right] = \Gamma(\eta)$$

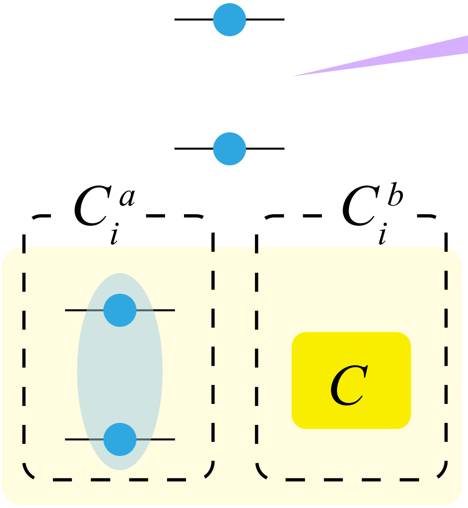
with $\eta' > \eta$.

(F. Ding, X. Hu, and H. Fan, Phys. Rev. A 103, 022403 (2021))

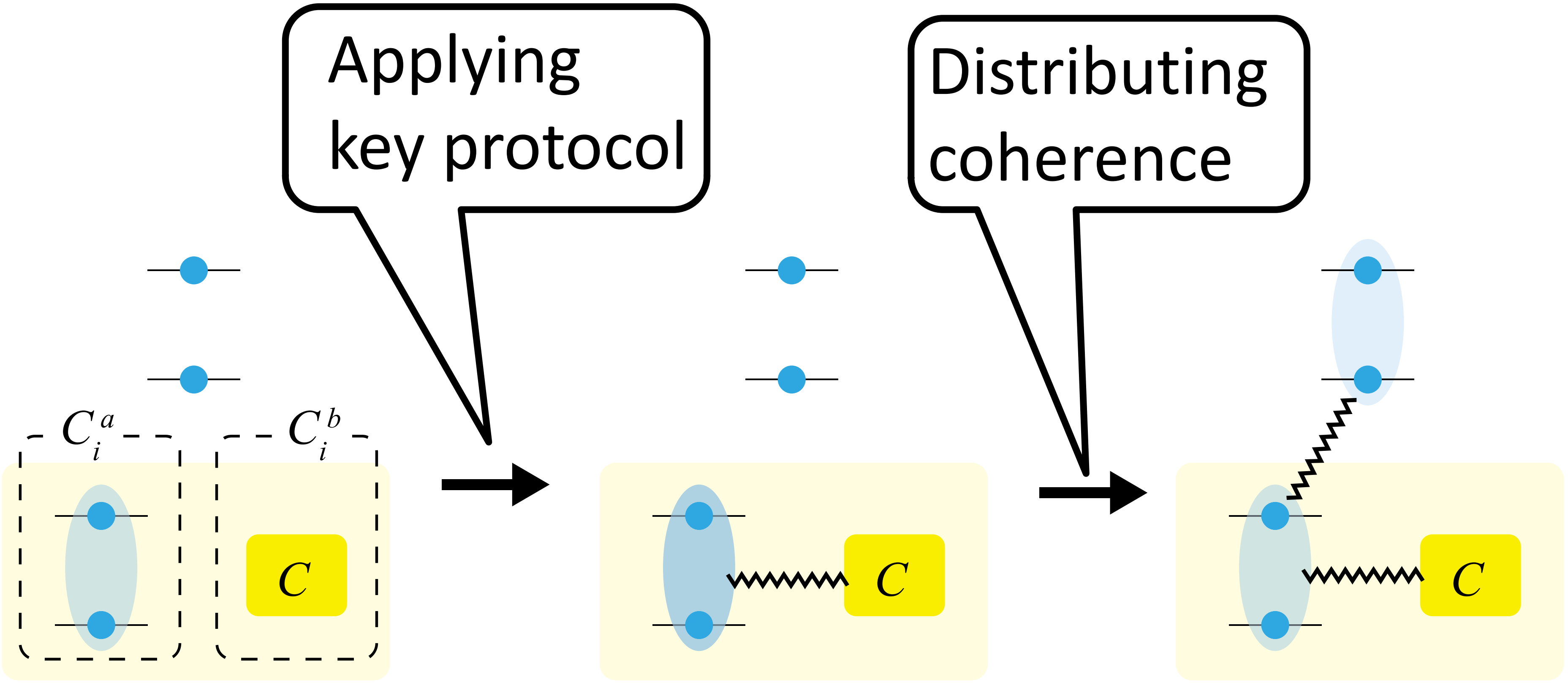


Step 1: from scratch to finite coherence

Incoherent state.
(We can discard this system finally)

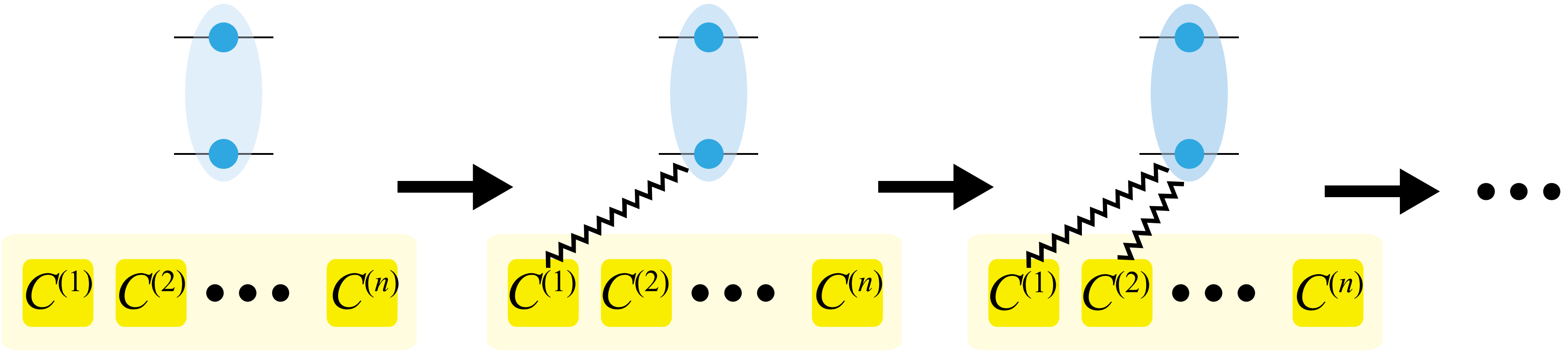


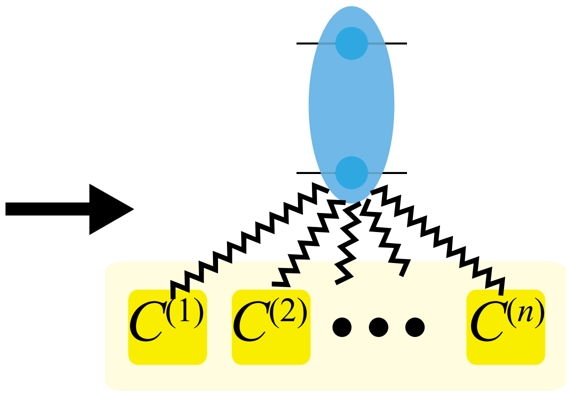
Step 1: from scratch to finite coherence



Step 2: Amplifying coherence

Applying key protocol many times,

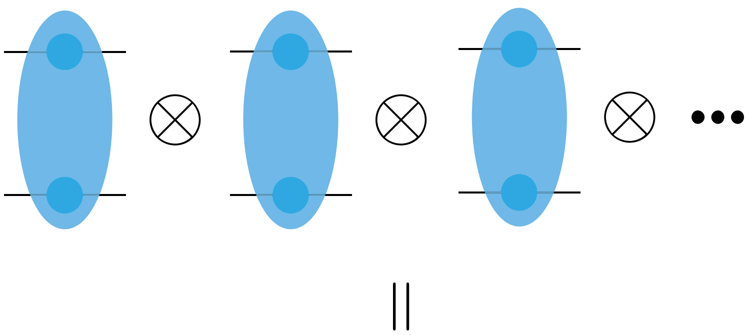




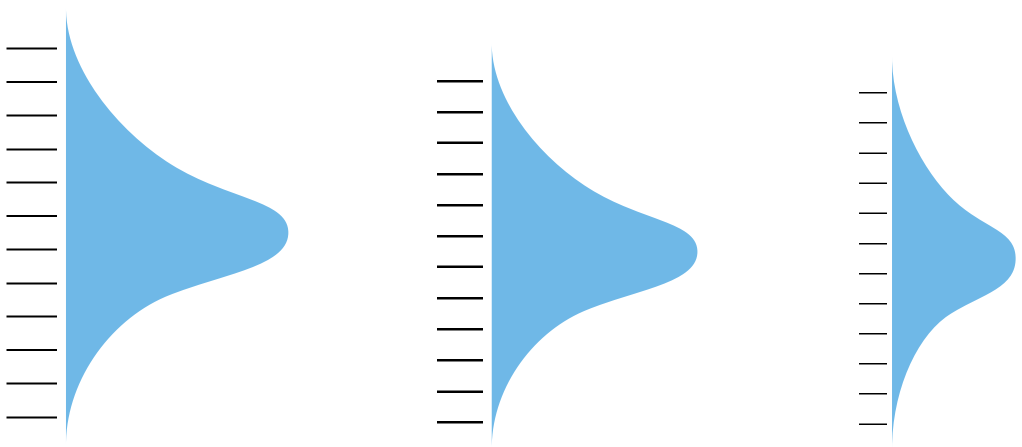
we obtain a state arbitrarily close to the maximally coherent state.

Step 3: approximated Gaussian state

Product state of many maximally coherent states is close to the Gaussian state on a ladder energy level.

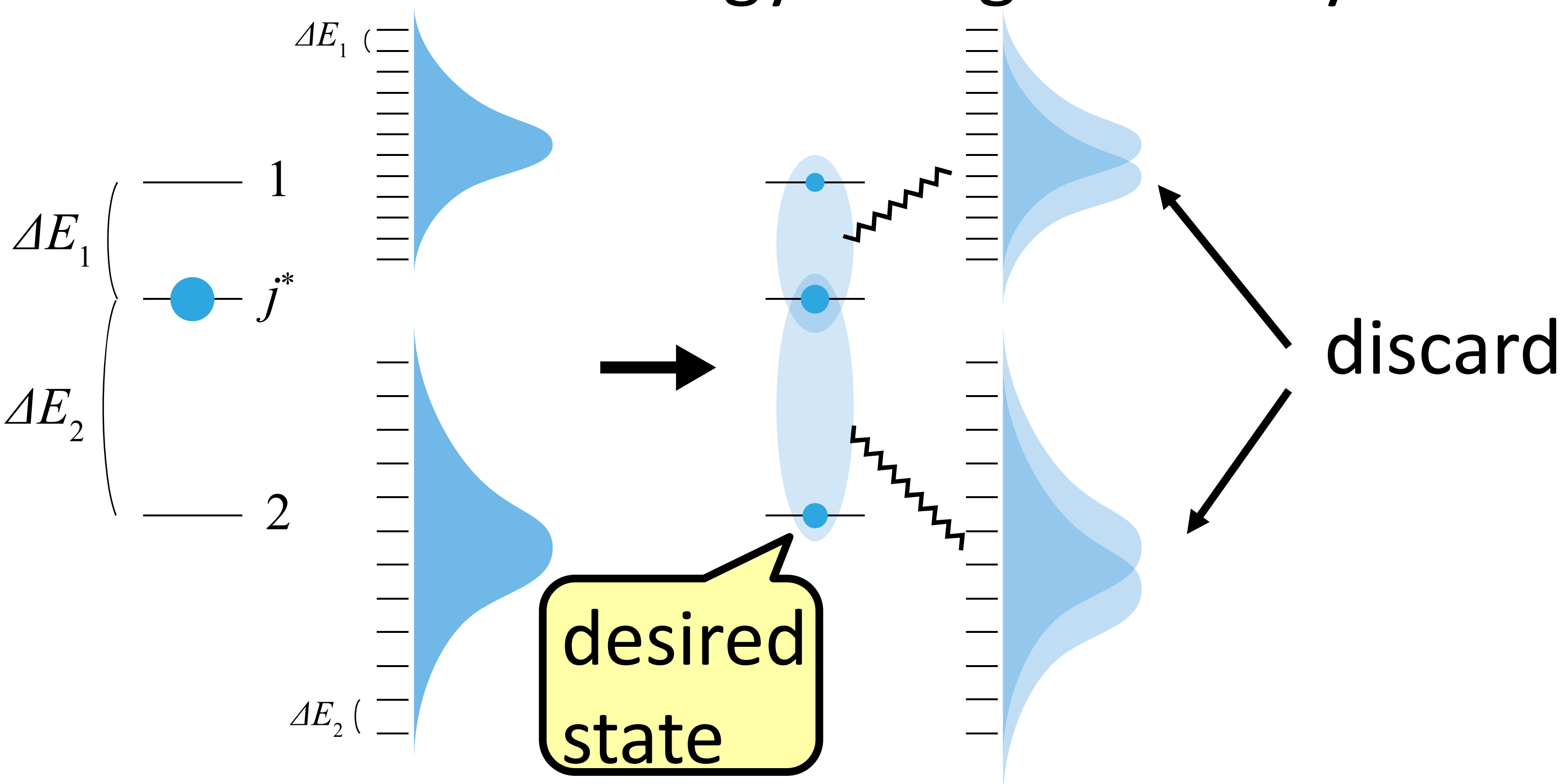


We can obtain many Gaussian states with arbitrary level spacing of ladder.



Step 4: emulating unitary operation

Prepare Gaussian states with all level spacing in the system to absorb the energy change in the system.



(related to Y. Aharonov and L. Susskind, Phys. Rev. 155, 1428 (1967), S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Rev. Mod. Phys. 79, 555 (2007).)

Symmetric map with correlated catalyst

In Step 1, correlation among catalysts is essential.

On the other hand, other steps might be realized in the case of correlated catalyst.

Conjecture (intuitive): If ρ has nonzero coherence, ρ can be converted to any ρ' via a symmetric map with correlated catalyst.

(R. Takagi and N. Shiraishi, arXiv:2106.12592)

Precise statement of conjecture

Let $I(\rho)$ be a set of pairs (i, j) where $\rho_{ij} \neq 0$.

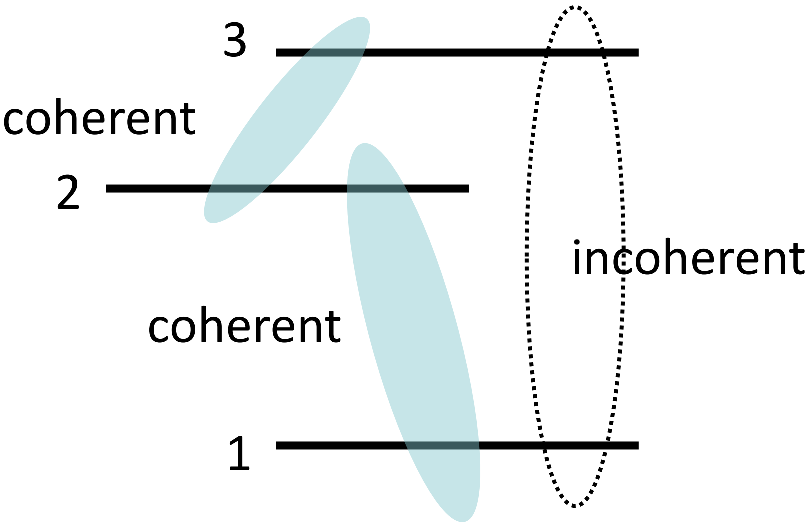
Let $J(\rho)$ be a set of pairs (i, j) such that $E_i - E_j$ can be written as a linear combination of integer multiple of energy difference in $I(\rho)$:

$$E_i - E_j = \sum_{(k,l)\in I(\rho)} c_{kl}(E_k - E_l), \qquad c_{kl} \in \mathbb{Z}$$

<u>Conjecture</u>:
ρ is transformable to ρ' via energy conserving map with correlated catalyst if and only if $I(\rho') \subset J(\rho)$.

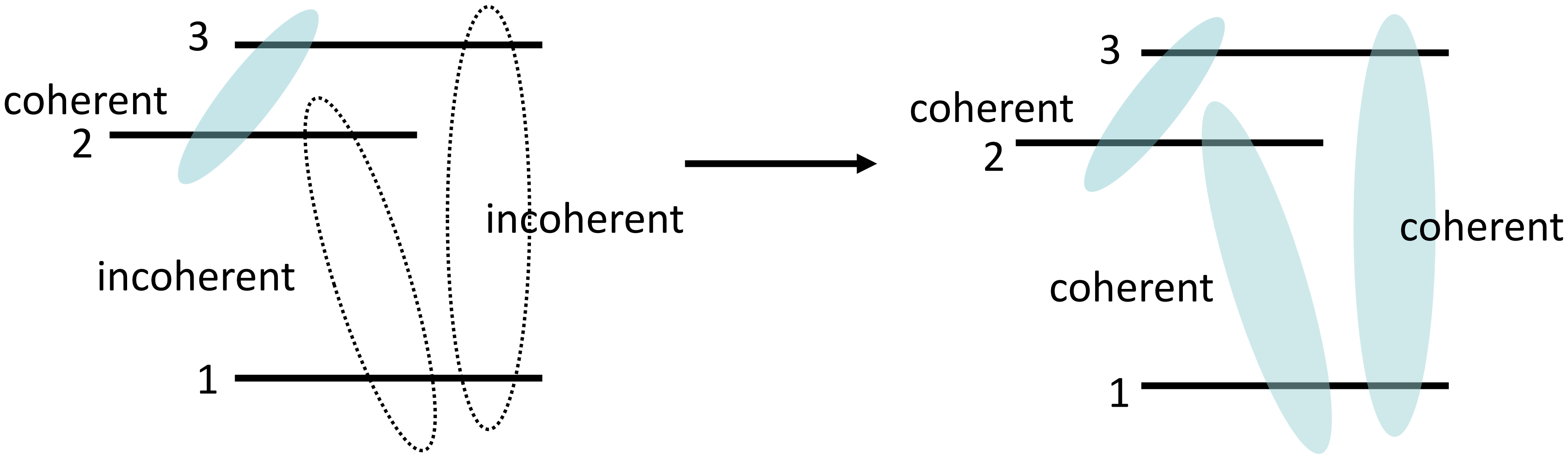
Remarks on conjecture 1

If 1-2 and 2-3 have coherence, we can create coherence between 1-3 by correlated catalyst even if $E_2 - E_1$ and $E_3 - E_2$ are irrational with each other.



$$\rho = \begin{pmatrix} a & d & 0 \\ d & b & e \\ 0 & e & c \end{pmatrix} \longrightarrow \rho' = \begin{pmatrix} a' & d' & f' \\ d' & b' & e' \\ f' & e' & c' \end{pmatrix}$$

Remarks on conjecture 2



If $E_2 - E_1 = m(E_3 - E_2)$ (m: integer), coherence between 1-2 can create coherence between 2-3 and 1-3.

Summary

- In small quantum systems, the nec.&suff. condition for conversion $\rho \to \rho'$ via GPM with correlated catalyst is given by the second law:
  $$\boldsymbol{F(\rho) \geq F(\rho')}$$
- The proof technique can be extended to other problems in resource theories.
- Resource theory of asymmetry with marginal catalysts is trivial.

END